

# A SIMPLE PROOF OF THE MARTINGALE REPRESENTATION THEOREM USING NONSTANDARD ANALYSIS

TRISTRAM DE PIRO

ABSTRACT. We give a proof of the Martingale Representation Theorem using the methods of nonstandard analysis. We use it to find a general method of solving stochastic differential equations.

We introduce the following spaces;

**Definition 0.1.** *Let  $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$ , and set  $\eta = 2^\nu$ . Define;*

$$\overline{\Omega}_\eta = \overline{\mathcal{T}}_\nu = \{x \in {}^*\mathcal{R} : 0 \leq x < 1\}$$

*We let  $\mathcal{C}_\eta$  consist of internal unions of the intervals  $[\frac{i}{\eta}, \frac{i+1}{\eta})$ , for  $0 \leq i \leq \eta - 1$ , and let  $\mathcal{D}_\nu$  consist of internal unions  $[\frac{i}{\nu}, \frac{i+1}{\nu})$ , for  $0 \leq i \leq \nu - 1$ .*

*We define counting measures  $\mu_\eta$  and  $\lambda_\nu$  on  $\mathcal{C}_\eta$  and  $\mathcal{D}_\nu$  respectively, by setting  $\mu_\eta([\frac{i}{\eta}, \frac{i+1}{\eta})) = \frac{1}{\eta}$  and  $\lambda_\nu([\frac{i}{\nu}, \frac{i+1}{\nu})) = \frac{1}{\nu}$ .*

*We let  $(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$  and  $(\overline{\mathcal{T}}_\nu, \mathcal{D}_\nu, \lambda_\nu)$  be the resulting \*-finite measure spaces, in the sense of [2], and let  $(\overline{\Omega}_\eta, L(\mathcal{C}_\eta), L(\mu_\eta))$ ,  $(\overline{\mathcal{T}}_\nu, L(\mathcal{D}_\nu), L(\lambda_\nu))$  be the associated Loeb spaces.*

*We let  $V(\mathcal{C}_\eta) = \{f : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}, f(x) = f(\frac{[ \eta x ]}{\eta})\}$  and  $W(\mathcal{C}_\eta) \subset V(\mathcal{C}_\eta)$  be the set of measurable functions  $f : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$ , with respect to  $\mathcal{C}_\eta$ , in the sense of [2]. Then  $W(\mathcal{C}_\eta)$  is a \*-finite vector space over  ${}^*\mathcal{C}$ , of dimension  $\eta$ , <sup>(1)</sup>.*

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<sup>1</sup> By a \*-vector space, one means an internal set  $V$ , for which the operations  $+: V \times V \rightarrow V$  of addition and scalar multiplication  $\cdot : {}^*\mathcal{C} \times V \rightarrow V$  are internal. Such spaces have the property that \*-finite linear combinations  ${}^*\sum_{i \in I} \lambda_i \cdot v_i$ ,  $(*)$ , for a \*-finite index set  $I$ , belong to  $V$ , by transfer of the corresponding standard result for vector spaces. We say that  $V$  is a \*-finite vector space, if there exists a \*-finite index set  $I$  and elements  $\{v_i : i \in I\}$  such that every  $v \in V$  can be written as a combination  $(*)$ , and the elements  $\{v_i : i \in I\}$  are independent, in the sense that

**Definition 0.2.** Given  $n \in \mathcal{N}_{>0}$ , we let  $\Omega_n = \{m \in \mathcal{N} : 0 \leq m < 2^n\}$ , and let  $C_n$  be the set of sequences of length  $n$ , consisting of 1's and -1's. We let  $\theta_n : \Omega_n \rightarrow \mathcal{N}^n$  be the map which associates  $m \in \Omega_n$  with its binary representation, and let  $\phi_n : \Omega_n \rightarrow C_n$  be the composition  $\phi_n = (\gamma \circ \theta_n)$ , where, for  $\bar{m} \in \mathcal{N}^n$ ,  $\gamma(\bar{m}) = 2.\bar{m} - \bar{1}$ . For  $\nu \in {}^*\mathcal{N} \setminus \mathcal{N}$ , we let  $\phi_\nu : \Omega_\nu \rightarrow C_\nu$  be the map, obtained by transfer of  $\phi_n$ , which associates  $i \in {}^*\mathcal{N}$ ,  $0 \leq i < 2^\nu$ , with an internal sequence of length  $\nu$ , consisting of 1's and -1's. Similarly, for  $\eta = 2^\nu$ , we let  $\psi_\eta : \overline{\Omega_\eta} \rightarrow C_\nu$  be defined by  $\psi_\eta(x) = \phi_\nu([\eta x])$ . For  $1 \leq j \leq \nu$ , we let  $\omega_j : C_\nu \rightarrow \{1, -1\}$  be the internal projection map onto the  $j$ 'th coordinate, and let  $\omega_j : \overline{\Omega_\eta} \rightarrow \{1, -1\}$  also denote the composition  $(\omega_j \circ \psi_\eta)$ , so that  $\omega_j \in W(\overline{\Omega_\eta})$ . By convention, we set  $\omega_0 = 1$ . For an internal sequence  $\bar{t} \in C_\nu$ , we let  $\omega_{\bar{t}} : \overline{\Omega_\eta} \rightarrow \{1, -1\}$  be the internal function defined by;

$$\omega_{\bar{t}} = \prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}$$

Again, it is clear that  $\omega_{\bar{t}} \in W(\overline{\Omega_\eta})$ .

**Lemma 0.3.** The functions  $\{\omega_j : 1 \leq j \leq \nu\}$  are  $*$ -independent in the sense of [1], (Definition 19), in particular they are orthogonal with respect to the measure  $\mu_\eta$ . Moreover, the functions  $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$  form an orthogonal basis of  $V(\overline{\Omega_\eta})$ , and, if  $\bar{t} \neq \overline{-1}$ ,  $E_\eta(\omega_{\bar{t}}) = 0$ , and  $\text{Var}_\eta(\omega_{\bar{t}}) = 1$ , where,  $E_\eta$  and  $\text{Var}_\eta$  are the expectation and variance corresponding to the measure  $\mu_\eta$ .

*Proof.* According to the definition, we need to verify that for an internal index set  $J = \{j_1, \dots, j_s\} \subseteq \{1, \dots, \nu\}$ , and an internal tuple  $(\alpha_1, \dots, \alpha_s)$ , where  $s = |J|$ ;

$$\begin{aligned} & \mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \dots, \omega_{j_k}(x) < \alpha_k, \dots, \omega_{j_s}(x) < \alpha_s) \\ &= \prod_{k=1}^s \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) \quad (*) \end{aligned}$$

Without loss of generality, we can assume that each  $\alpha_{j_k} > -1$ , as if some  $\alpha_{j_k} \leq -1$ , both sides of  $(*)$  are equal to zero. Let  $J' = \{j' \in J : -1 < \alpha_{j'} \leq 1\}$  and  $J'' = \{j'' \in J : 1 < \alpha_{j''}\}$ , so  $J = J' \cup J''$ . Then;

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if  $(*) = 0$ , then each  $\lambda_i = 0$ . It is clear, by transfer of the corresponding result for finite dimensional vector space over  $\mathcal{C}$ , that  $V$  has a well defined dimension given by  $\text{Card}(I)$ , see [3], even though  $V$  may be infinite dimensional, considered as a standard vector space.

$$\begin{aligned}
& \mu_\eta(x : \omega_{j_1}(x) < \alpha_1, \dots, \omega_{j_s}(x) < \alpha_s) \\
&= \frac{1}{\eta} \text{Card}(z \in C_\nu : z(j') = -1 \text{ for } j' \in J', z(j'') \in \{-1, 1\} \text{ for } j'' \in J'') \\
&= \frac{1}{2^\nu} \text{Card}(z \in C_\nu : z(j') = -1 \text{ for } j' \in J') = \frac{2^{\nu-s'}}{2^\nu} = 2^{-s'}
\end{aligned}$$

where  $s' = \text{Card}(J')$ . Moreover;

$$\prod_{k=1}^s \mu_\eta(x : \omega_{j_k}(x) < \alpha_k) = \prod_{j' \in J'} \mu_\eta(x : \omega_{j'}(x) = -1) = 2^{-s'}$$

as  $\mu_\eta(x : \omega_j(x) = -1) = \frac{1}{2}$ , for  $1 \leq j \leq \nu$ . Hence, (\*) is shown. That \*-independence implies orthogonality follows easily by transfer, from the corresponding fact, for finite measure spaces, that  $E(X_{j_1}X_{j_2}) = E(X_{j_1})E(X_{j_2})$ , for the standard expectation  $E$  and independent random variables  $\{X_{j_1}, X_{j_2}\}$ , (\*\*). Hence, by (\*\*);

$$E_\eta(\omega_{j_1}\omega_{j_2}) = E_\eta(\omega_{j_1})E_\eta(\omega_{j_2}) = 0, (j_1 \neq j_2) (***)$$

as clearly  $E_\eta(\omega_j) = 0$ , for  $1 \leq j \leq \nu$ . If  $\bar{t} \neq \overline{-1}$ , let  $J' = \{j' : 1 \leq j' \leq \nu, \bar{t}(j') = 1\}$ , then;

$$E_\eta(\omega_{\bar{t}}) = E_\eta(\prod_{1 \leq j \leq \nu} \omega_j^{\frac{\bar{t}(j)+1}{2}}) = E_\eta(\prod_{j' \in J'} \omega_{j'}) = \prod_{j' \in J'} E_\eta(\omega_{j'}) = 0 (\#)$$

where, in (#), we have used the facts that  $J' \neq \emptyset$  and internal, and a simple generalisation of (\* \* \*), by transfer from the corresponding fact for finite measure spaces. Hence,  $1 = \omega_{\overline{-1}}$  is orthogonal to  $\omega_{\bar{t}}$ , for  $\bar{t} \neq \overline{-1}$ . If  $\bar{t}_1 \neq \bar{t}_2$  are both distinct from  $\overline{-1}$ , then, if  $J_1 = \{j : 1 \leq j \leq \nu, \bar{t}_1(j) = 1\}$  and  $J_2 = \{j : 1 \leq j \leq \nu, \bar{t}_2(j) = 1\}$ , so  $J_1 \neq J_2$  and  $J_1, J_2 \neq \emptyset$ , we have;

$$\begin{aligned}
& E_\eta(\omega_{\bar{t}_1}\omega_{\bar{t}_2}) \\
&= E_\eta(\prod_{j \in J_1} \omega_j \cdot \prod_{j \in J_2} \omega_j) (\#\#) \\
&= E_\eta(\prod_{j \in (J_1 \setminus J_2)} \omega_j \cdot \prod_{j \in (J_2 \setminus J_1)} \omega_j) (\#\#\#) \\
&= E_\eta(\prod_{j \in (J_1 \setminus J_2)} \omega_j) E_\eta(\prod_{j \in (J_2 \setminus J_1)} \omega_j) = 0 (\#\#\#\#)
\end{aligned}$$

In (# #), we have used the definition of  $J_1$  and  $J_2$ , and in (# # #), we have used the fact that  $(J_1 \cup J_2) = (J_1 \cap J_2) \sqcup (J_1 \setminus J_2) \sqcup (J_2 \setminus J_1)$ , and  $\omega_j^2 = 1$ , for  $1 \leq j \leq \nu$ . Finally, in (# # # #), we have used the facts that  $(J_1 \setminus J_2)$

and  $(J_2 \setminus J_1)$  are disjoint, and at least one of these sets is nonempty, the result of (#) and a similar generalisation of (\*\*). This shows that the functions  $\{\omega_{\bar{t}} : \bar{t} \in C_\nu\}$  are orthogonal, (\*\*). That they form a basis for  $V(\overline{\Omega_\eta})$  follows immediately, by transfer, from (\*\*) and the corresponding fact for finite dimensional vector spaces. The final calculation is left to the reader.  $\square$

We require the following;

**Definition 0.4.** For  $0 \leq l \leq \nu$ , we define  $\sim'_l$ , on  $C_\nu$ , to be the internal equivalence relation given by;

$$\bar{t}_1 \sim'_l \bar{t}_2 \text{ iff } \bar{t}_1(j) = \bar{t}_2(j) \ (\forall j \leq l)$$

We extend this to an internal equivalence relation on  $\overline{\Omega_\eta}$ , which we denote by  $\sim_l$ ;

$$x_1 \sim_l x_2 \text{ iff } \psi_\eta(x_1) \sim_l \psi_\eta(x_2) \ (*)$$

We let  $\mathcal{C}_\eta^l$  be the  $*$ -finite algebra generated by the partition of  $\overline{\Omega_\eta}$  into the  $2^l$  equivalence classes with respect to  $\sim_l, (*)$ . As is easily verified, we have  $\mathcal{C}_\eta^{l_1} \subseteq \mathcal{C}_\eta^{l_2}$ , if  $l_1 \leq l_2$ ,  $\mathcal{C}_\eta^0 = \{\emptyset, \overline{\Omega_\eta}\}$  and  $\mathcal{C}_\eta = \mathcal{C}_\eta^\nu$ . For  $0 \leq l \leq \nu$ , we let  $W(\mathcal{C}_\eta^l) \subseteq W(\mathcal{C}_\eta)$  be the set of measurable functions  $f : \overline{\Omega_\eta} \rightarrow {}^*\mathcal{C}$ , with respect to  $\mathcal{C}_\eta^l$ . We will refer to the collection  $\{\mathcal{C}_\eta^l : 0 \leq l \leq \nu\}$  of  $*$ -finite algebras, as the nonstandard filtration associated to  $\overline{\Omega_\eta}$ . We produce a standard filtration  $\{\mathfrak{D}_t : t \in [0, 1]\}$ , (\*\*), by following the method of [1], see Definition 7.14 of [3], (replacing the equivalence relation  $\sim$  there, by  $\sim_l$ , as given in (\*), and being careful to use the index  $\nu$  instead of  $\eta$ . Note that Lemma 7.15 of [3] still applies in this case.) We also require a slight modification of the construction of Brownian motion in [1]. Namely, we take;

$$\chi(t, x) = \frac{1}{\sqrt{\nu}} ({}^*\sum_{i=1}^{[\nu t]} \omega_i)$$

$$\text{and } W(t, x) = {}^\circ\chi(t, x), \ (t, x) \in [0, 1] \times \overline{\Omega_\eta} \ (**).$$

One of the advantages of the non-standard approach to stochastic calculus, is that it allows one to show easily that every stochastic integral is a martingale. We follow the notation from Chapter 7 of [3], again using the filtration (\*\*) of Definition 0.4 to replace the one from Definition 7.14, and its subsequent applications;

**Theorem 0.5.** *If  $g \in \mathcal{G}_0$ , and  $f$  is a 2-lifting of  $g$ , then  $I(t, x)$ , as in Definition 7.20 of [3], is equivalent, as a stochastic process, to a martingale, with respect to the filtration  $\mathfrak{D}_t$ , <sup>(2)</sup>.*

*Proof.* Let  $I'$  be the modification of  $I$ , as given in the proof of Theorem 7.25 of [3]. Then  $I'$  and agree  $I$  on  $[0, 1] \times C$ , where  $P(C) = 1$ , and  $P = L(\mu_\eta)$ , so they are equivalent as stochastic processes. We show that  $I'$  is a martingale.

(i) follows from the fact that  $I$  is  $\mathfrak{B} \times \mathfrak{D}$  measurable, and  $I = I'$  a.e  $\mu \times L(\mu_\eta)$ , (\*). Here, completeness of the product is required.

(ii). By the construction in the proof of Theorem 7.25 of [3],  $I'_t$  is measurable with respect to  $\mathfrak{D}'_t \subset \mathfrak{D}_t$ .

(iii). We have, for  $t \in [0, 1]$ ;

$$\begin{aligned} \int_{\overline{\Omega_\eta}} I'^2(t, x) dL(\mu_\eta) &= \int_{\overline{\Omega_\eta}} I^2(t, x) dL(\mu_\eta) \\ &= \int_{\overline{\Omega_\eta}} \circ F^2(t, x) d\mu_\eta \\ &\leq \circ \int_{\overline{\Omega_\eta}} F^2(t, x) d\mu_\eta \\ &= \circ \int_{\overline{\Omega_\eta}} \int_0^t f^2(t, x) d\lambda_\nu d\mu_\eta = \|g\|_{L^2([0, t] \times \overline{\Omega_\eta})}^2 \quad (\dagger) \end{aligned}$$

using (\*), Definition 7.20, (see notation in Theorem 7.24), Theorem 3.16 and the proof of Theorem 7.22 in [3]. Hence  $I'_t \in L^2(\overline{\Omega_\eta}, \mathcal{C}_\eta, P)$ , so  $I'_t \in L^1(\overline{\Omega_\eta}, \mathcal{C}_\eta, P)$ , by Holder's inequality, see [4].

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<sup>2</sup> By which I mean a function  $I : [0, 1] \times \overline{\Omega_\eta} \rightarrow \mathcal{R}$ , such that;

(i).  $I$  is  $\mathfrak{B} \times \mathfrak{D}$  measurable (complete product).

(ii).  $I_t$  is measurable with respect to  $\mathfrak{D}_t$ , for  $t \in [0, 1]$ .

(iii).  $E(|I_t|) < \infty$ , for  $t \in [0, 1]$ .

(iv).  $E(I_t | \mathfrak{D}_s) = I_s$ , if  $s < t$  belong to  $[0, 1]$ .

(v). For  $C \subset \overline{\Omega_\eta}$ , with  $L(\mu_\eta)(C) = 1$ , and  $x \in C$ , the paths  $\gamma_x : [0, 1] \rightarrow \mathcal{R}$ , where  $\gamma_x(t) = I(t, x)$ , are continuous.

Most of this definition can be found in [5], see also [6] for a thorough discussion of discrete time martingales.

(iv). Suppose  $s < t$ . We first show that  $E(I'_t|\mathfrak{D}'_s) = I'_s$ , ( $\dagger\dagger$ ). Suppose  $i \in {}^*\mathcal{N}$ , with  $\frac{i}{\nu} \simeq s$ , then we claim that  $E(I'_t|\sigma(\mathcal{C}_\eta^i)^{comp}) = I'_s$ , ( $**$ ). As  $I_t = I'_t$  a.e  $P$ , we have  $E(I'_t|\sigma(\mathcal{C}_\eta^i)^{comp}) = E(I_t|\sigma(\mathcal{C}_\eta^i)^{comp})$ . We can also see that  $F_t \in SL^2(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$ . This follows from the calculation ( $\dagger$ ), Theorem 3.34(i) of [3], and the fact that;

$$\int_{\overline{\Omega}_\eta} I^2(t, x) dL(\mu_\eta) = \|g\|_{L^2([0, t] \times \overline{\Omega}_\eta)}^2$$

by Ito's isometry, as  $g \in \mathcal{G}_0$ . Hence, by Theorem 3.34(iv) of [3],  $F_t \in SL^1(\overline{\Omega}_\eta, \mathcal{C}_\eta, \mu_\eta)$ , ( $***$ ). Applying Theorem 7.3(ii) of [3] and ( $***$ );

$$E(I_t|\sigma(\mathcal{C}_\eta^i)^{comp}) = E({}^\circ F_t|\sigma(\mathcal{C}_\eta^i)^{comp}) = {}^\circ E(F_t|\mathcal{C}_\eta^i)$$

We have;

$$E(F_t|\mathcal{C}_\eta^i) = \sum_{j=0}^{i-1} f(\frac{j}{\nu}, x) \frac{\omega_{j+1}}{\sqrt{\nu}}$$

by \*-independence of the sequence  $\{\omega_j\}_{0 \leq j \leq [\nu t]+1}$ . Letting  $s' = \frac{i-1}{\nu}$ , so  $s' \simeq s$ ,  $E(F_t|\mathcal{C}_\eta^i) = F_{s'}$ . We have, using Theorem 7.24 of [3], that  $I_s = I_{s'}$  a.e  $P$ , so  $I'_s = I_s = I_{s'}$  a.e  $P$ . As  $I'_s$  is  $\sigma(\mathcal{C}_\eta^i)^{comp}$ -measurable, we have  $E(I'_t|\sigma(\mathcal{C}_\eta^i)^{comp}) = I'_s$ , showing ( $**$ ). As  $\mathfrak{D}'_s \subset \sigma(\mathcal{C}_\eta^i)^{comp}$ , and  $I'_s$  is  $\mathfrak{D}'_s$ -measurable, we have  $E(I'_t|\mathfrak{D}'_s) = I'_s$ , showing ( $\dagger\dagger$ ).

If  $A \in \mathfrak{D}_s$ , then, by Lemma 7.15(i) of [3],  $A \in \mathfrak{D}'_{s_1}$ , for  $s < s_1 < t$ . As  $E(I'_t|\mathfrak{D}'_{s_1}) = I'_{s_1}$ , to show (iv), it is sufficient to prove that;

$$\int_A I'_s dL(\mu_\eta) = \lim_{s_1 \rightarrow s} \int_A I'_{s_1} dL(\mu_\eta) \quad (\dagger\dagger\dagger)$$

To show ( $\dagger\dagger\dagger$ ), observe that  $\|I'_{s_1} - I'_s\|_2^2 \leq \|g_{[0, s_1]} - g_{[0, s]}\|_2^2$  by ( $\dagger$ ), where  $g_{[0, s_1]}$  is obtained by truncating the function  $g$  to the interval  $[0, s_1]$ , ( $^3$ ). Using Holder's inequality and the DCT, we have  $\lim_{s_1 \rightarrow s} \|I'_{s_1} - I'_s\|_1 \leq \lim_{s_1 \rightarrow s} \|g_{[0, s_1]} - g_{[0, s]}\|_1 = 0$ . Therefore, ( $\dagger\dagger\dagger$ ) is shown. This proves (iv).

(v). This is Theorem 25 of [1].

□

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<sup>3</sup>Technically, you need to show that  $I_{s_1}$  is the non standard stochastic integral of  $g_{[0, s_1]}$ , and then apply Theorem 7.22 of [3], however, this is clear by truncating the corresponding lift of  $g$ .

We proceed to show the converse, that every martingale can be represented as a stochastic integral, using the nonstandard approach.

**Lemma 0.6.** *For  $0 \leq l \leq \nu$ , a basis of the  $*$ -finite vector space  $W(\mathcal{C}_\eta^l)$  is given by  $D_l = \bigcup_{0 \leq m \leq l} B_m$ , where, for  $1 \leq m \leq \nu$ ,  $B_m = \{\omega_{\bar{t}} : \bar{t}(m) = 1, \bar{t}(m') = -1, m < m' \leq \nu\}$ , and  $B_0 = \{\omega_{\bar{1}}\}$ .*

*Proof.* The case when  $l = 0$  is clear as  $\omega_{\bar{1}} = 1$ , and using the description of  $\mathcal{C}_\eta^0$  in Definition 0.4. Using the observation (\*) there, we have, for  $1 \leq l \leq \nu$ , that  $W(\mathcal{C}_\eta^l)$  is a  $*$ -finite vector space of dimension  $2^l$ . Using Lemma 0.3, and the fact that  $\text{Card}(D_l) = 2^l$ , it is sufficient to show each  $\omega_{\bar{t}} \in D_l$  is measurable with respect to  $\mathcal{C}_\eta^l$ . We have that, for  $1 \leq j \leq l$ ,  $\omega_j$  is measurable with respect to  $\mathcal{C}_\eta^j \subseteq \mathcal{C}_\eta^l$ . Hence, the result follows easily, by transfer of the result for finite measure spaces, that the product  $X_{j_1} X_{j_2}$ , of two measurable random variables  $X_{j_1}$  and  $X_{j_2}$  is measurable.  $\square$

**Definition 0.7.** *We define a nonstandard martingale to be a  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable function  $Y : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$ , such that;*

- (i). *For  $t \in \overline{\mathcal{T}}_\nu$ ,  $Y_{\lfloor \frac{\nu t}{\nu} \rfloor}$  is measurable with respect to  $\mathcal{C}_\eta^{[\nu t]}$ .*
- (ii).  *$E_\eta(Y_{\lfloor \frac{\nu t}{\nu} \rfloor} | \mathcal{C}_\eta^{[\nu s]}) = Y_{\lfloor \frac{\nu s}{\nu} \rfloor}$ , for  $(0 \leq s \leq t < 1)$ , <sup>(4)</sup>.*
- (iii).  *$E_\eta(|Y_{\lfloor \frac{\nu t}{\nu} \rfloor}|)$  is finite.*

**Lemma 0.8.** *Let  $Y : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$  be a  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable function, satisfying (i) and (ii) of Definition 0.7, then;*

$$Y_t(x) = \sum_{j=0}^{[\nu t]} c_j(t, x) \omega_j(x)$$

*where, if  $s = \frac{j}{\nu}$ ,  $c_j : [s, 1) \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{C}$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable and  $c_j(s, x) = c_j(t, x)$ , for  $0 \leq s \leq t < 1$ .*

*Proof.* Using (ii), we have that  $E_\eta(Y_t) = E_\eta(Y_t | \mathcal{C}_\eta^0) = Y_0$ . Replacing  $Y_t$  by  $Y_t - Y_0$ , we can, without loss of generality, assume that  $E_\eta(Y_t) = 0$ , for  $t \in [0, 1)$ . By (i) and Lemma 0.6;

$$Y_t = \sum_{j=1}^{[\nu t]} c_j(t, x) \omega_j(x)$$

where;

$$c_j(t, x) = \sum_{a=1}^{j-1} \sum_{i_1 < \dots < i_a; 1}^{j-1} p_j^{(i_1, \dots, i_a)}(t) \omega_{i_1} \dots \omega_{i_a}(x)$$

Again, using (ii), and the fact that  $c_{\nu t'}(t, x) \omega_{\nu t'}$  is orthogonal to  $c_{\nu s'}(t, x) \omega_{\nu s'}$ , for  $s' \leq s < t' \leq t$ , (\*), we have;

$$\sum_{j=1}^{[\nu s]} c_j(t, x) \omega_j(x) = \sum_{j=1}^{[\nu s]} c_j(s, x) \omega_j(x)$$

Equating coefficients, and using the fact that  $D_{[\nu t]}$  is a basis for  $W(\mathcal{C}_\eta^{[\nu t]})$ , we obtain  $c_j(s, x) = c_j(t, x)$ , for all  $0 \leq s \leq t < 1$ .  $\square$

**Lemma 0.9.** *Let  $X \in L^2(\overline{\Omega}_\eta, L(\mu_\eta))$ , with  $\|X\|_{L^2(\overline{\Omega}_\eta, L(\mu_\eta))} = c < \infty$ , then there exists  $\overline{X} \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$ , with  ${}^\circ \overline{X} = X$ , a.e  $dL(\mu_\eta)$  and  $\|\overline{X}\|_{SL^2(\overline{\Omega}_\eta, \mu_\eta)} \simeq c$ , and a sequence  $\{\overline{X}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\overline{\Omega}_\eta, \mu_\eta)$ , such that;*

$$(i). \quad \overline{X}_1 = \overline{X}.$$

$$(ii). \quad E_\eta(\overline{X}_1 | \mathcal{C}_\eta^i) = \overline{X}_{\frac{i}{\nu}}, \text{ for } (0 \leq i \leq \nu)$$

$$(iii). \quad E_\eta((\overline{X}_{\frac{d}{\nu}} - \overline{X}_{\frac{c}{\nu}})^2 (\overline{X}_{\frac{b}{\nu}} - \overline{X}_{\frac{a}{\nu}})^2)$$

$$= E_\eta((\overline{X}_{\frac{d}{\nu}} - \overline{X}_{\frac{c}{\nu}})^2) E_\eta((\overline{X}_{\frac{b}{\nu}} - \overline{X}_{\frac{a}{\nu}})^2), \text{ for } (0 \leq a < b \leq c < d \leq \nu)$$

*Proof.* Suppose  $\overline{Y} \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$ , then, using Lemma 0.3, we can write  $\overline{Y} = * \sum_{j=0}^{2^\nu-1} \lambda_{\tilde{t}_j} \omega_{\tilde{t}_j}$ . <sup>(5.)</sup> We first show how to satisfy the conditions (i), (ii). Letting  $\overline{Y}_{\frac{i}{\nu}} = * \sum_{j=0}^{2^i-1} \lambda_{i, \tilde{t}_j} \omega_{\tilde{t}_j}$ , for  $0 \leq i \leq \nu - 1$ , we have;

$$E_\eta(\overline{Y} | \mathcal{C}_\eta^i) = \overline{Y}_{\frac{i}{\nu}}$$

$$\text{iff } E_\eta(\overline{Y} - \overline{Y}_{\frac{i}{\nu}} | \mathcal{C}_\eta^i) = 0$$

$$\text{iff } E_\eta((\overline{Y} - \overline{Y}_{\frac{i}{\nu}}) \omega_{\tilde{t}_k}) = 0, \text{ for } 0 \leq k \leq 2^i - 1$$

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<sup>5</sup>We order  $C_\nu$  inductively as follows. If  $\tilde{t} \in C_\nu$ , we define  $length(\tilde{t}) = \max(I_{\tilde{t}})$ , where  $I_{\tilde{t}} = \{j : \tilde{t}(j) = 1\}$ . Suppose we have ordered  $C_{j,\nu} \subset C_\nu$ , for some  $0 \leq j \leq \nu$ , where  $C_{j,\nu} = \{\tilde{t} \in C_\nu : length(\tilde{t}) \leq j\}$ . If  $\tilde{t} \in C_{j+1,\nu}$ , we can write  $\tilde{t}$  as  $\tilde{s} \omega_{j+1}$ , where  $\tilde{s} \in C_{j,\nu}$ . Then, we define an ordering on  $C_{j+1,\nu}$  by  $\tilde{t}_1 \leq \tilde{t}_2$  if  $length(\tilde{t}_1) < length(\tilde{t}_2)$ , or if  $length(\tilde{t}_1) = length(\tilde{t}_2) = j + 1$ ,  $length(\tilde{s}_1) \leq \tilde{s}_2$



$$\text{iff } E_\eta((\sum_{j=0}^{2^i-1} (\lambda_{\bar{t}_j} - \lambda_{i,\bar{t}_j}) \omega_{\bar{t}_j} + \sum_{j=2^i}^{2^\nu-1} \lambda_{\bar{t}_j} \omega_{\bar{t}_j}) \omega_{\bar{t}_k}) = 0,$$

$$\text{for } 0 \leq k \leq 2^i - 1$$

$$\text{iff } \lambda_{\bar{t}_j} = \lambda_{i,\bar{t}_j}, \text{ for } 0 \leq j \leq 2^i - 1. \quad (*)$$

Letting  $\bar{Y} = \bar{Y}_1$ , from the definition of  $S$ -integrability and nonstandard conditional expectation, see Definition 3.17 and Definition 7.1 of [3], it follows that the sequence  $\{\bar{Y}_{\frac{i}{\nu}} : 0 \leq i \leq \nu\} \subset SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , and conditions (i), (ii) are satisfied. Now, we consider condition (iii). We have that;

$$\begin{aligned} & (\bar{X}_{\frac{b}{\nu}} - \bar{X}_{\frac{a}{\nu}}) \\ &= \sum_{j=0}^{2^a-1} (\lambda_{b,\bar{t}_j} - \lambda_{a,\bar{t}_j}) \omega_{\bar{t}_j} + \sum_{j=2^a}^{2^b-1} \lambda_{b,\bar{t}_j} \omega_{\bar{t}_j} \\ &= \sum_{j=a}^{2^b-1} \lambda_{\bar{t}_j} \omega_{\bar{t}_j}, \text{ using } (*) \\ & E_\eta((\bar{X}_{\frac{b}{\nu}} - \bar{X}_{\frac{a}{\nu}})^2) \\ &= (\sum_{j=2^a}^{2^b-1} \lambda_{\bar{t}_j}^2), \text{ by Lemma 0.3} \\ & E_\eta((\bar{X}_{\frac{d}{\nu}} - \bar{X}_{\frac{c}{\nu}})^2) E_\eta((\bar{X}_{\frac{b}{\nu}} - \bar{X}_{\frac{a}{\nu}})^2) \\ &= (\sum_{j=2^c}^{2^d-1} \lambda_{\bar{t}_j}^2) (\sum_{j=2^a}^{2^b-1} \lambda_{\bar{t}_j}^2) \\ & (\bar{X}_{\frac{d}{\nu}} - \bar{X}_{\frac{c}{\nu}})^2 (\bar{X}_{\frac{b}{\nu}} - \bar{X}_{\frac{a}{\nu}})^2 \\ &= \sum_{j,k=2^c,j',k'=2^a}^{2^d-1,2^b-1} \lambda_{\bar{t}_j} \lambda_{\bar{t}_k} \lambda_{\bar{t}_{j'}} \lambda_{\bar{t}_{k'}} \omega_{\bar{t}_j} \omega_{\bar{t}_k} \omega_{\bar{t}_{j'}} \omega_{\bar{t}_{k'}} \\ & E_\eta((\bar{X}_{\frac{d}{\nu}} - \bar{X}_{\frac{c}{\nu}})^2 (\bar{X}_{\frac{b}{\nu}} - \bar{X}_{\frac{a}{\nu}})^2) \\ &= \sum_{j=2^c,j'=2^a}^{2^d-1,2^b-1} \lambda_{\bar{t}_j}^2 \lambda_{\bar{t}_{j'}}^2 + \sum_{(j,k,j',k') \in W_{a,b,c,d}} \lambda_{\bar{t}_j} \lambda_{\bar{t}_k} \lambda_{\bar{t}_{j'}} \lambda_{\bar{t}_{k'}} \end{aligned}$$

where;

$$W_{a,b,c,d} = \{(j, k, j', k') \in {}^*\mathcal{Z}^4 : \{j, k\} \subset [2^c, 2^d - 1], \{j', k'\} \subset$$

$$[2^a, 2^b - 1], j \neq k, j' \neq k', \omega_{\bar{t}_j} \omega_{\bar{t}_k} \omega_{\bar{t}_{j'}} \omega_{\bar{t}_{k'}} = 1\}$$

The condition (iii) is then given by the  $*$ -algebraic equations;

$$^* \sum_{(j,k,j',k') \in W_{a,b,c,d}} \lambda_{\bar{t}_j} \lambda_{\bar{t}_k} \lambda_{\bar{t}_{j'}} \lambda_{\bar{t}_{k'}} = 0 \quad (0 \leq a < b \leq c < d)$$

Working over  $V(\mathcal{C}_\eta)$ , we thus obtain at most  $\kappa = \nu^4$   $*$ -algebraic equations for the unknown variables  $\{\lambda_{\bar{t}_j} : 0 \leq j \leq 2^\nu - 1\}$ . Let  $V_{inc} \subset {}^*\mathcal{C}^{2^\nu}$  denote the corresponding  $*$ -algebraic variety of dimension  $2^\nu - \tau$ , where  $0 \leq \tau \leq \kappa \leq \nu^4$ , with  $*$  $\mathcal{R}$ -coefficients. By transfer of the corresponding result over  $\mathcal{C}$ , there exists a projection  $pr_{2^\nu, 2^\nu - \tau} : {}^*\mathcal{C}^{2^\nu} \rightarrow {}^*\mathcal{C}^{2^\nu - \tau}$ , with respect to the canonical orthonormal basis  $\{e_j = \sqrt{\nu}e'_j : 0 \leq j \leq 2^\nu - 1\}$  of  $(V(\mathcal{C}_\eta), \langle \cdot, \cdot \rangle_{SL^2})$ ,<sup>(6)</sup> such that the image  $pr_{2^\nu, 2^\nu - \tau}(V_{inc}) \subset {}^*\mathcal{C}^{2^\nu - \tau}$  is  $*$ -Zariski dense, (\*\*). Now, let  $X \in L^2(\bar{\Omega}_\eta, L(\mu_\eta))$  be given, with  $\|X\|_{L^2} = c < \infty$ . Using Theorem 3.13 of [3], we can find  $\bar{X}_1 \in SL^2(\bar{\Omega}_\eta, L(\mu_\eta))$ , such that  ${}^\circ\bar{X}_1 = X$  a.e  $dL(\mu_\eta)$ , and  $\|\bar{X}_1\|_{SL^2} \simeq c$ , (#). Truncating  $X_1$ , we can assume that, for any given infinite  $\theta$ ,  $|X_1| \leq \theta$ , and (#) still holds. Let  $w_1 = \{\lambda_{1, \bar{t}_j} : 0 \leq j \leq 2^\nu - 1\}$  denote the corresponding vector in  $V(\bar{\Omega}_\eta)$ ,  $v_2 = pr_{2^\nu, 2^\nu - \tau}(w_1)$ , and  $w_2 = (v_2, \bar{0})$ . For the corresponding  $\bar{X}_2 \in V(\bar{\Omega})$ , we still have that  ${}^\circ\bar{X}_2 = X$ , a.e  $dL(\mu_\eta)$ , as if  $I = \{i : 0 \leq i < 2^\nu - 1, e_i \in Ker(pr_{2^\nu, 2^\nu - \tau})\}$ , and  $S = \bigcup_{i \in I} [\frac{i}{2^\nu}, \frac{i+1}{2^\nu})$ , then  $\mu_\eta(S) = \frac{\tau}{2^\nu - 1} \leq \frac{\nu^4}{2^\nu - 1} \simeq 0$  and  $L(\mu_\eta)(S) = 0$ . Moreover,  $\bar{X}_2 \in SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , and  $\|\bar{X}_2\|_{SL^2} \simeq c$ , as we have redefined  $X_1$  on a set  $S$  of infinitesimal measure, and choosing  $\theta = \nu$ , we have  $\theta\mu_\eta(S) = \frac{\nu^5}{2^\nu - 1} \simeq 0$ . Now, using (\*\*), we can find  $v_3$ , with  $|v_3(i) - v_2(i)| \leq \frac{1}{4^\nu}$ , for  $i \in I$ , such that  $v_3 \in (pr_{2^\nu, 2^\nu - \tau}(V_{inc}) \cap {}^*\mathcal{R}^{2^\nu - \tau})$ . Again, if  $w_3 = (v_3, 0)$ , for the corresponding  $\bar{X}_3 \in V(\bar{\Omega}_\eta)$ , we have that  ${}^\circ\bar{X}_3 = X$  a.e  $L(\mu_\eta)$ , as, for  $x \in (\bar{\Omega}_\eta \setminus S)$ ,  $X_3(x) \simeq X_2(x)$ , and  $\bar{X}_3 \in SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , with  $\|\bar{X}_3\|_{SL^2} \simeq c$ , as, for  $x \in (\bar{\Omega}_\eta \setminus S)$ , we have  $|\bar{X}_3(x) - \bar{X}_2(x)| \leq \frac{\sqrt{\nu}}{4^\nu} \simeq 0$ , and  $\|\bar{X}_3 - \bar{X}_2\|_{SL^2} \leq \frac{1}{4^\nu}(2^\nu - \tau) \leq \frac{1}{2^\nu} \simeq 0$ . ..... By the above, we have that  $\|\bar{X}_3\|_{L^2(\bar{\Omega}_\eta, L(\mu_\eta))} \leq c + 1$ . Now, let  $U_{pinc} = pr_{2^\nu, 2^\nu - \tau}(V_{inc})$ , which, without loss of generality, we can assume to be open in  ${}^*\mathcal{R}^{2^\nu - \tau}$ . Let  $V_{inc} \cap (U_{pinc} \times {}^*\mathcal{R}^\tau)$ , be defined by the set of equations  $\{p_i(\bar{x}, \bar{y}) : 1 \leq i \leq \kappa + 1\}$ , where  $l(\bar{x}) = 2^\nu - \tau$ , and  $l(\bar{y}) = \tau$ . For  $1 \leq j \leq \tau$ , we consider the equations;

$$q_j(v_3, y_j) = \exists y_1 \dots y_{j-1} y_{j+1} \dots y_\tau (\bigwedge_{1 \leq i \leq \kappa+1} p_i(v_3, y_j))$$

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<sup>6</sup>Here, the inner product on  $V(\mathcal{C}_\eta)$  is given by  $\langle f, g \rangle = \int_{\bar{\Omega}_\eta} f(x) \bar{g}(x) d\mu_\eta(x)$ , and  $e'_j : \bar{\Omega}_\eta \rightarrow {}^*\mathcal{C}$  is the  $\mathcal{C}_\eta$ -measurable function, given by  $e'_j(\frac{i}{\nu}) = \delta_{ij}$ , for  $0 \leq i \leq \nu - 1$ .

Reducing the varieties  $\{K_j : 1 \leq j \leq \tau\}$  defined by these equations, we obtain  $*$ -polynomials  $s_j(v_3, y_4)$ , such that the solution set  $W \subset V_{inc}$ . We now obtain solutions  $(v_3, v_{4,j})$  with bounded complexity. Observe that, if  $v_{3,i}$  is a coordinate of  $v_3$ , for some  $i \in I$ , then;

$$\|v_{3,i}e_i\|_{SL^2}^2 = \int_{[\frac{i}{2^\nu}, \frac{i+1}{2^\nu})} \overline{X}_3^2 d\mu_\eta \leq \frac{\theta}{2^\nu} = \frac{\nu}{2^\nu} \simeq 0$$

Then, for each polynomial  $s_j(v_3, y_{4,j})$ , for any  $i \in I$ , we can find a solution  $v_{4,j}$ , with  $\|v_{4,j}e_j\|_{SL^2} \leq \|v_{3,i}e_i\| \leq \frac{\sqrt{\nu}}{2^{\frac{\nu}{2}}}$ , <sup>(7)</sup>. Then, if  $v_4$  has coordinates  $v_{4,j}$ , for  $j \in (2^\nu \setminus I)$ , we have  $(v_3, v_4) \in V_{inc}$ , <sup>(##)</sup> and;

$\|v_4\|_{SL^2}^2 = * \sum_{1 \leq j \leq \tau} \|v_{4,j}\|^2 \leq \tau \frac{\sqrt{\nu}}{2^{\frac{\nu}{2}}} \leq \nu^4 \frac{\sqrt{\nu}}{2^{\frac{\nu}{2}}} = \epsilon \simeq 0$ . It then follows, that for the corresponding  $\overline{X}_4 \in V(\overline{\Omega}_\eta)$ , that we have  ${}^\circ \overline{X}_4 = X$  a.e  $dL(\mu_\eta)$ , as  $\mu_\eta(S) \simeq 0$ , condition (i). Moreover,  $\overline{X}_4 \in SL^2(\overline{\Omega}_\eta, \mu_\eta)$ , as;

$$\int_S (\overline{X}_4 - \overline{X}_3)^2 d\mu_\eta = \|v_4\|_{SL^2}^2 \leq \epsilon \simeq 0$$

$$\text{and } \overline{X}_4|_{\overline{\Omega}_\eta \setminus S} = \overline{X}_3|_{\overline{\Omega}_\eta \setminus S}$$

We obtain condition (iii) from <sup>(##)</sup>. Letting  $\{\overline{X}_{\frac{4,i}{\nu}} = E_\eta(\overline{X}_4 | \mathcal{C}_\eta^i) : 0 \leq i \leq \nu\}$ , we obtain condition (ii).  $\square$

**Lemma 0.10.** *Let  $X$  be a martingale, see footnote 2 for the definition, then there exists a nonstandard martingale  $\overline{X}$ , see Definition 0.7, with  ${}^\circ(\overline{X}_t) = X_{\circ t}$ , for  $t \in \overline{\mathcal{T}}_\nu$ , a.e  $L(\mu_\eta)$ .*

*Proof.* By (i) of footnote 2, we have  $X$  is  $\mathfrak{B} \times \mathfrak{D}$ -measurable. We claim that  $X \in L^1([0, 1] \times \overline{\Omega}_\eta)$ , <sup>(\*)</sup>. Without loss of generality, we can assume that  $X \geq 0$ , <sup>(8)</sup> Then <sup>(\*)</sup> follows from the fact that, for  $0 \leq t \leq 1$ ,

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<sup>7</sup>This follows, by transfer, from the result, that, for any polynomial  $t(y)$  of degree at most  $n$ , with coefficients  $0 \leq c_i \leq a$ ,  $0 \leq i \leq n$ , for any solution  $y_i$ ,  $1 \leq i \leq n$ ,  $0 \leq y_i \leq a$ . In order to see this, let  $A$  be the  $n \times n$  matrix, defined by  $A_{ij} = \delta_{ij} y_i$ , for  $1 \leq i, j \leq n$ . We have that  $|y_i| \leq \lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}}$ , where  $\|\cdot\|$  is the spectral norm, given by  $\max_{1 \leq i \leq n} \lambda_i$ , where, for  $1 \leq i \leq n$ ,  $\lambda_i$  is an eigenvalue of  $AA^*$ . We have,  $\|A^k\| \leq (\sum_{i=1}^n y_i^{2k})^{\frac{1}{2}}$ , so  $y_i \leq \lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}} \leq (\sum_{i=1}^n y_i^{2k})^{\frac{1}{2k}} \leq (\sum_{i=1}^n y_i)^{2k \cdot \frac{1}{k}} = \sum_{i=1}^n y_i \leq a$ .

<sup>8</sup>In order to see this, it is sufficient to show that  $X^+$  is a martingale, <sup>(\*)</sup>. We have  $X = X^+ - X^-$ , and, by (iv), for  $0 \leq t \leq 1$ ;

$$X_t = X_t^+ - X_t^- = E(X_1 | \mathfrak{D}_t) = E(X_1^+ - X_1^- | \mathfrak{D}_t) = Y_t - Y_t' \quad (**)$$

where  $Y_t = E(X_1^+ | \mathfrak{D}_t)$  and  $Y_t' = E(X_1^- | \mathfrak{D}_t)$ . It follows easily, modifying  $Y$  to  $Y^1$ , and  $Y'$  to  $Y'^1$ , a.e  $L(\lambda_\nu) \times L(\mu_\eta)$ , if necessary, and, using the tower law

$E(X_t) = E(X_t|\mathfrak{D}_0) = X_0$ , by (iv) of 2, and so;

$$\int_{[0,1] \times \overline{\Omega}_\eta} X(t, x) d(L(\lambda_\nu) \times L(\mu_\eta)) = X_0 < \infty$$

by (iii) of footnote 2 and Fubini's theorem, see [4]. Using Theorem 3.13 of [3], see also [1], we can find  $X^2 \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$ , with  $(^\circ X^2) = X_1$ , a.e  $L(\mu_\eta)$ , ( $\dagger$ ). We now define  $\overline{X} : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$  by taking  $\overline{X}(t, x) = (E_\eta(X^2|\mathcal{C}_\eta^{[\nu t]}))(x)$ . We may assume that  $\overline{X}$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta$  measurable, by the definition of  $E_\eta(|)$ , see footnote 25 of Chapter 7, [3], and transfer of the corresponding result for finite measure spaces. Then, by Theorem 7.3 of [3];

$$({}^\circ \overline{X})(t, x) = {}^\circ(E_\eta(X^2|\mathcal{C}_\eta^{[\nu t]}))(x) = E({}^\circ X^2|\sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}) \quad (**)$$

Moreover, if  $A \in \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}$ , we have;

$$\int_A X_{\circ t} dL(\mu_\eta) = \lim_{t' \rightarrow \circ t} \int_A X_{t'} dL(\mu_\eta) = \int_A X_1 dL(\mu_\eta) \quad (***)$$

using (iv), (v) of footnote 2 and the result of (\*) to apply the DCT. Hence, as  $\mathfrak{D}_{\circ t} \subset \sigma(\mathcal{C}_\eta^{[\nu t]})^{comp} \subset \mathfrak{D}_{\nu'}$ , for  $0 \leq \circ t < t'$ , using (\*\*) in Definition 0.4, we have;

$$E({}^\circ X^2|\sigma(\mathcal{C}_\eta^{[\nu t]})^{comp}) = E({}^\circ X^2|\mathfrak{D}_{\circ t}) = E(X_1|\mathfrak{D}_{\circ t}) = X_{\circ t}$$

by (\*\*), ( $\dagger$ ) and (iv) of footnote 2. By (\*\*), we then have  $({}^\circ \overline{X}_t) = X_{\circ t}$ , a.e  $L(\mu_\eta)$ . We now verify conditions (i), (ii), (iii) of Definition 0.7. (i) is clear by Definition of  $\overline{X}$  and footnote 25 of Chapter 7, [3]. (ii) follows by transfer of the tower law for the conditional expectation  $E_\eta(|)$ , see again footnote 25 of Chapter 7. (iii) follows immediately from the fact that  $X^2 \in SL^1(\overline{\Omega}_\eta, \mu_\eta)$ , and the definition of  $E_\eta(|)$ .  $\square$

**Definition 0.11.** Let  $\overline{X}$  be as in Lemma 0.7, and let  $\{c_j(t, x) : 1 \leq j \leq \nu\}$  be given as in Lemma 0.8. Then we define  $\overline{H} : \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$  and  $\overline{Z} : \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$  by;

$$\overline{H}(t, x) = \sqrt{\nu} c_j(s, x)$$

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and definition of conditional expectations, see [6], that  $Y, Y'$  are martingales and  $Y, Y' \geq 0$ . We then have, by (\*\*), that  $X_t^+ = Y_t$  and  $X_t^- = Y_t'$  a.e  $L(\mu_\eta)$ . Hence, (\*) is shown.

where  $t = \frac{j}{\nu}$  and  $s \geq t$ , and;

$$\overline{Z}(x) = {}^* \sum_{0 \leq j \leq \nu-2} (\overline{X}_{\frac{j+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2$$

**Lemma 0.12.** *If  $X$  is a martingale in the sense of footnote 2, with independent increments, with the additional requirement that  $X_1 \in L^2(\overline{\Omega}_\eta, L(\mu_\eta))$ , then  $\overline{H} \in SL^2(\mathcal{T}_\nu \times \overline{\Omega}_\eta, \lambda_\nu \times \mu_\eta) \subset SL^1(\mathcal{T}_\nu \times \overline{\Omega}_\eta, \lambda_\nu \times \mu_\eta)$  and  $\overline{Z} \in SL^2(\overline{\Omega}_\eta, \mu_\eta) \subset SL^1(\overline{\Omega}_\eta, \mu_\eta)$ , if we choose  $X^2$  from Lemma 0.10 to be in  $SL^2(\overline{\Omega}_\eta, \mu_\eta)$ . (need to show that the lift  $\overline{X}$  has independent increments.)*

*Proof.* We first verify condition (i) of Definition 3.17 in [3], for  $\overline{H}$ . We have;

$$\begin{aligned} & \int_{\mathcal{T}_\nu \times \overline{\Omega}_\eta} |\overline{H}(t, x)|^2 d\lambda_\nu d\mu_\eta \\ &= \frac{1}{\nu} {}^* \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} |\overline{H}(\frac{j}{\nu}, x)|^2 d\mu_\eta \\ &= \frac{1}{\nu} {}^* \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} \nu |c_j(1, x)|^2 d\mu_\eta \quad (\dagger) \\ &= {}^* \sum_{0 \leq j \leq \nu-1} \int_{\overline{\Omega}_\eta} |c_j(1, x)|^2 d\mu_\eta \\ &= \int_{\overline{\Omega}_\eta} |\overline{X}_1|^2 d\mu_\eta \quad (\dagger\dagger) \end{aligned}$$

where, in  $(\dagger)$ , we have used Definition 0.11, and, in  $(\dagger\dagger)$ , we have used the fact that  $\overline{X}_1 = {}^* \sum_{0 \leq j \leq \nu-1} c_j(1, x) \omega_j$ , by Lemma 0.8, and the orthogonality observation  $(*)$  there. This gives (i) by the assumption of the Lemma. We now consider finite martingales  $\{V^n : n \in \mathcal{N}\}$  on finite measure spaces  $\Omega_e$ , <sup>(9)</sup>. Without loss of generality, we assume that  $E_e(V_1^n) = V_0^n = 0$ . We let  $W_n = \sum_{0 \leq j \leq n-1} (V_{\frac{j+1}{n}}^n - V_{\frac{j}{n}}^n)^2$ , then;

$$\begin{aligned} E_e(W_n) &= \sum_{0 \leq j \leq n-1} \int_{\Omega_e} (V_{\frac{j+1}{n}}^n - V_{\frac{j}{n}}^n)^2 d\mu_n \\ &= \int_{\Omega_e} (V_1^n)^2 d\mu_n = c_{n,e} \quad (*) \end{aligned}$$

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<sup>9</sup> By which I mean, for  $n \in \mathcal{N}$ ,  $e = 2^n$ , a  $\mathcal{D}_n \times \mathcal{C}_e$ -measurable function  $V^n : \mathcal{T}_n \times \Omega_e \rightarrow \mathcal{R}$ , where  $\{\mathcal{D}_n, \mathcal{C}_e, \mathcal{T}_n, \Omega_e\}$  are as in Definition 0.1, with the obvious modification to finite  $n \in \mathcal{N}$ , satisfying the axioms of Definition 0.7, again adapting the filtration in Definition 0.4 to  $\{\mathcal{C}_e^l : 0 \leq l \leq n\}$ , for finite  $n \in \mathcal{N}$ , and altering the (conditional) expectation to  $E_e$ , for finite  $e \in \mathcal{N}$ .

where, in (\*), we have used the fact that the increments  $(V_{\frac{j+1}{n}}^n - V_{\frac{j}{n}}^n)$  are orthogonal with respect to the measure  $\mu_n$ , (using footnote 9, Definition 0.7(ii)), Pythagoras's Theorem and the identity  $V_1^n = \sum_{0 \leq j \leq n-1} (V_{\frac{j+1}{n}}^n - V_{\frac{j}{n}}^n)$ . We let;

$$d_{j,n,e} = E((V_{\frac{j+1}{n}}^n - V_{\frac{j}{n}}^n)^2), \text{ for } 0 \leq j \leq n-1$$

and;

$$J_{n,e} = \{j \in \mathcal{Z} : 0 \leq j \leq n-1, d_{j,n,e} \geq \frac{c_{n,e}}{\sqrt{n}}\}$$

$$K_{n,e} = \{j \in \mathcal{Z} : 0 \leq j \leq n-1, j \notin J_{n,e}\}$$

$$b_{n,e} = \text{Card}(J_{n,e})$$

$$R_{n,e} = \{(t, x) \in \mathcal{T}_n \times \Omega_e : [nt] \in J_{n,e}\}$$

$$S_{n,e} = (\mathcal{T}_n \times \Omega_e \setminus R_{n,e})$$

Observing, by (\*), that  $\sum_{0 \leq j \leq n-1} d_{j,n,e} = c_{n,e}$ , we have  $b_{n,e} \frac{c_{n,e}}{\sqrt{n}} \leq c_{n,e}$ , hence  $b_{n,e} \leq \sqrt{n}$ . Similarly, defining  $\{d_{j,\nu,\eta}, J_{\nu,\eta}, K_{\nu,\eta}, b_{\nu,\eta}, c_{\nu,\eta}, R_{\nu,\eta}, S_{\nu,\eta}\}$  for the nonstandard martingale  $\bar{X}$ , we obtain  $^* \sum_{0 \leq j \leq \nu-1} d_{j,\nu,\eta} = c_{\nu,\eta}$  and  $b_{\nu,\eta} \leq \sqrt{\nu}$ , (\*\*). We have;

$$\begin{aligned} & \int_{R_{\nu,\eta}} |\bar{H}(t, x)| d\mu_\eta d\lambda_\nu \\ &= \frac{1}{\nu} {}^* \sum_{j \in J_{\nu,\eta}} \int_{\bar{\Omega}_\eta} \sqrt{\nu} |c_j(1, x)| d\mu_\eta \\ &\leq (\frac{1}{\nu} b_{\nu,\eta} \sqrt{\nu}) \max_{j \in J_{\nu,\eta}} \int_{\bar{\Omega}_\eta} |c_j(1, x)| d\mu_\eta \\ &\leq \max_{j \in J_{\nu,\eta}} E_\eta(|c_j(1, x)|) \simeq 0 \text{ (***)} \end{aligned}$$

where we have used the definitions of  $\bar{H}$ ,  $J_{\nu,\eta}$  and  $b_{\nu,\eta}$ , the bound in (\*\*), and the fact that  $E_\eta(|c_j(1, x)|) \simeq 0$ , for  $0 \leq j \leq \nu-1$ , (\*\*\*). This follows easily from Lemma 0.10, and the fact that;

$$|c_j(1, x)| = |c_j(1, x)\omega_j(x)| = |\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x)|$$

using Lemma 0.8, and Theorem 3.4(i) of [3]. We let;

$$T(x) = {}^* \sum_{j \in K_{\nu, \eta}} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2$$

Then;

$$E_{\eta}(T) = {}^* \sum_{j \in K_{\nu, \eta}} \int_{\bar{\Omega}_{\eta}} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2 d\mu_{\eta} = f_{\nu, \eta}$$

$$\leq {}^* \sum_{0 \leq j \leq \nu-1} \int_{\bar{\Omega}_{\eta}} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2 d\mu_{\eta} = c_{\nu, \eta}$$

as in (\*), and;

$$Var_{\eta}(T) = {}^* \sum_{j \in K_{\nu, \eta}} Var_{\eta}((\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2)$$

We let;

$$\bar{Z}(x) = S(x) = {}^* \sum_{0 \leq j \leq \nu-2} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2$$

$$\text{Then } E_{\eta}(S) = {}^* \sum_{0 \leq j \leq \nu-2} \int_{\bar{\Omega}_{\eta}} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2 d\mu_{\eta} = c_{\nu, \eta}$$

and;

$$\begin{aligned} Var_{\eta}(S) &= E_{\eta}(({}^* \sum_{0 \leq j \leq \nu-2} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2 - c_{\nu, \eta})^2) \\ &= E_{\eta}({}^* \sum_{0 \leq j \leq \nu-2} (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^4 \\ &\quad + {}^* \sum_{0 \leq i \neq j \leq \nu-2} (\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^2 (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2 \\ &\quad - 2c_{\nu, \eta} {}^* \sum_{0 \leq i \leq \nu-2} (\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^2 + c_{\nu, \eta}^2) \\ &= {}^* \sum_{0 \leq j \leq \nu-2} E_{\eta}((\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^4) \\ &\quad + {}^* \sum_{0 \leq i \neq j \leq \nu-2} E_{\eta}((\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^2 (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2) \\ &\quad - 2c_{\nu, \eta} {}^* \sum_{0 \leq i \leq \nu-2} E_{\eta}((\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^2) + c_{\nu, \eta}^2 \\ &= {}^* \sum_{0 \leq i \leq \nu-2} E_{\eta}((\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^4) - c_{\nu, \eta}^2 \\ &\quad + {}^* \sum_{0 \leq i \neq j \leq \nu-2} E_{\eta}((\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^2 (\bar{X}_{\frac{j+1}{\nu}}(x) - \bar{X}_{\frac{j}{\nu}}(x))^2) \\ &= {}^* \sum_{0 \leq i \leq \nu-2} E_{\eta}((\bar{X}_{\frac{i+1}{\nu}}(x) - \bar{X}_{\frac{i}{\nu}}(x))^4) - ({}^* \sum_{0 \leq i \leq \nu-1} d_{i, \nu, \eta})^2 \end{aligned}$$

$$\begin{aligned}
& +^* \sum_{0 \leq i \neq j \leq \nu-2} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2 (\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2) \\
& =^* \sum_{0 \leq i \neq j \leq \nu-2} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^4) - d_{i,\nu,\eta}^2 \\
& +^* \sum_{0 \leq i \neq j \leq \nu-2} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2 (\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2) - d_{i,\nu,\eta} d_{j,\nu,\eta} \\
& =^* \sum_{0 \leq i \neq j \leq \nu-2} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^4) - d_{i,\nu,\eta}^2
\end{aligned}$$

by independence of the increments  $\{(\overline{X}_{\frac{i+1}{\nu}} - \overline{X}_{\frac{j}{\nu}}), (\overline{X}_{\frac{i+1}{\nu}} - \overline{X}_{\frac{j}{\nu}})\}$ , for  $0 \leq i \neq j \leq \nu-1$ . We have;

$$^* \sum_{0 \leq i \leq \nu-1} d_{i,\nu,\eta}^2 \leq (\max_{0 \leq i \leq \nu-1} \{d_{i,\nu,\eta}\}) c_{\nu,\eta} \simeq 0$$

by continuity of paths. We have;

$$|(\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))| \leq \epsilon$$

for  $\epsilon \in \mathcal{R}_{>0}$ . Therefore;

$$E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^4) \leq \epsilon^2 E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2)$$

and

$$\begin{aligned}
& ^* \sum_{0 \leq i \leq \nu-1} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^4) \\
& \leq \epsilon^{2*} \sum_{0 \leq i \leq \nu-1} E_\eta((\overline{X}_{\frac{i+1}{\nu}}(x) - \overline{X}_{\frac{j}{\nu}}(x))^2) = \epsilon^2 c_{\nu,\eta}
\end{aligned}$$

As  $\epsilon$  was arbitrary, we conclude that  $Var_\eta(S) \simeq 0$ . Therefore;

$$\begin{aligned}
& Var_\eta(\overline{Z}) \\
& = E_\eta((\overline{Z} - c_{\nu,\eta})^2) \\
& = E_\eta(\overline{Z}^2) - 2c_{\nu,\eta} E_\eta(\overline{Z}) + c_{\nu,\eta}^2 \\
& = E_\eta(\overline{Z}^2) - c_{\nu,\eta}^2 \simeq 0
\end{aligned}$$

and, hence,  $E_\eta(\overline{Z}^2) = q_{\nu,\eta}$  is finite. By the transfer of Holder's inequality;



$$E_\eta(|\bar{Z} - c_{\nu,\eta}|) \leq Var_\eta(\bar{Z}) \simeq 0$$

Hence, if  $A \subset \bar{\Omega}_\eta$ , with  $\mu_\eta(A) \simeq 0$ ;

$$\int_A |\bar{Z}| d\mu_\eta - \int_A c_{\nu,\eta} d\mu_\eta \leq \int_A |\bar{Z} - c_{\nu,\eta}| d\mu_\eta \simeq 0$$

so  $\int_A |\bar{Z}| d\mu_\eta \simeq 0$  and  $\bar{Z} \in SL^1(\bar{\Omega}_\eta, \mu_\eta)$ . Similarly, as  $Var_\eta(\bar{Z}) \simeq 0$ ;

$$\int_A \bar{Z}^2 d\mu_\eta - 2c_{\nu,\eta} \int_A \bar{Z} d\mu_\eta + \int_A c_{\nu,\eta}^2 d\mu_\eta \simeq 0$$

hence,  $\bar{Z} \in SL^2(\bar{\Omega}_\eta, \mu_\eta)$ , ( $\dagger$ ).

We now verify condition (ii) of Definition 3.17 in [3], for  $\bar{H}$ .

Case 1. Let  $A \subset \bar{\Omega}_\eta$ , with  $\mu_\eta(A) \simeq 0$ , then;

$$\begin{aligned} & \int_{A \times \bar{T}_\eta} H(t, x)^2 d\mu_\eta d\lambda_\nu \\ &= \frac{1}{\nu} * \sum_{0 \leq j \leq \nu-1} \int_A \nu c_j(1, x)^2 d\mu_\eta \\ &= \int_A * \sum_{0 \leq j \leq \nu-1} c_j(1, x)^2 d\mu_\eta \\ &= \int_A * \sum_{0 \leq j \leq \nu-1} (\bar{X}_{\frac{j+1}{\nu}} - \bar{X}_{\frac{j}{\nu}})^2 d\mu_\eta = \int_A \bar{Z}^2 d\mu_\eta \simeq 0 \end{aligned}$$

by ( $\dagger$ ).

Case 2. Let  $B \subset \bar{T}_\nu$ , with  $\lambda_\nu(B) \simeq 0$ . Assume first that  $B = [\frac{i}{\nu}, \frac{j}{\nu})$ , with  $0 \leq i < j < \nu$ , and  $(j - i) \leq \epsilon\nu$ , for some infinitesimal  $\epsilon$ . Then;

$$\begin{aligned} & \int_{\bar{\Omega}_\eta \times B} H(t, x)^2 d\lambda_\nu d\mu_\eta \\ &= \int_{\bar{\Omega}_\eta \times [0, \frac{i}{\nu})} H(t, x)^2 d\lambda_\nu d\mu_\eta - \int_{\bar{\Omega}_\eta \times [0, \frac{i}{\nu})} H(t, x)^2 d\lambda_\nu d\mu_\eta \quad (!) \end{aligned}$$

We have;

$$X(t, x) = \sum_{0 \leq j \leq t\nu} c_j(1, x) \omega_j(x)$$

$$X(t, x)^2 = \sum_{0 \leq j, k \leq t\nu} c_j(1, x) c_k(1, x) \omega_j(x) \omega_k(x) \quad (\#)$$

Then;

$$\begin{aligned}
& \int_{\overline{\Omega}_\eta} (X_t)^2(x) d\mu_\eta \\
&= \sum_{0 \leq j, k \leq [t\nu]} \int_{\overline{\Omega}_\eta} c_j(1, x) c_k(1, x) \omega_j \omega_k d\mu_\eta \text{ (using } (\#)) \\
&= \sum_{0 \leq j \leq [t\nu]} \int_{\overline{\Omega}_\eta} c_j^2(1, x) d\mu_\eta \text{ (using Lemma 0.8) } (\#\#) \\
& \int_{[0, t] \times \overline{\Omega}_\eta} H(s, x)^2(x) d\lambda_\nu d\mu_\eta \\
&= \frac{1}{\nu} \sum_{0 \leq j \leq [t\nu]} \int_{\overline{\Omega}_\eta} \nu c_j^2(1, x) d\mu_\eta \\
&= \int_{\overline{\Omega}_\eta} X_t^2(x) d\mu_\eta \text{ (by } (\#\#)) \text{ } (\#\#\#)
\end{aligned}$$

Then, by (!) and (\#\#\#);

$$\begin{aligned}
& \int_{\overline{\Omega}_\eta \times B} H(t, x)^2 d\lambda_\nu d\mu_\eta \\
&= \int_{\overline{\Omega}_\eta} (X_{\frac{i}{\nu}})^2 - (X_{\frac{i}{\nu}})^2 d\mu_\eta \simeq 0
\end{aligned}$$

by continuity of paths. Now suppose that  $B$  is not an interval. Let;

$I = \{i \in [0, \nu] \cap {}^*\mathcal{N} : \frac{i}{\nu} \in B\}$ ,  $I' = \{i \in I : (i-1) \notin I\}$ . Then  $I, I'$  are internal and  $Card(I') = p$ . Let us assume that  $I'$  is equally spaced, that is;

$$I' = \{i[\frac{\nu}{2p}] : 0 \leq i \leq p-1\} \text{ (!!)}$$

For  $i_j \in I'$ , with  $1 \leq j \leq p$ , we let  $i'_j$  be defined by  $\frac{i'_j}{\nu} \in B, i_j \leq i'_j < i_{j+1}$ , and  $i'_j + 1 \notin B$ . Then, clearly,  $B = \bigcup_{1 \leq j \leq p} [\frac{i_j}{\nu}, \frac{i'_j}{\nu})$ , and, we assume that  $\lambda_\nu([\frac{i_j}{\nu}, \frac{i'_j}{\nu})) \leq \frac{1}{4p}$ , (!!!)

For  $1 \leq j \leq p$ , we let  $f_j = {}^*\sum_{1 \leq k \leq j} (i'_k - i_k) + 1$ . Inductively, we define  $X'$  to be a  $\mathcal{C}_\nu \times \mathcal{D}_\eta$  measurable function, on  $[0, 1) \times \overline{\Omega}_\eta$ , by;

$$\begin{aligned}
X'(\frac{[t\nu]}{\nu}, x) &= X(\frac{[t\nu]}{\nu}, x), t \in [0, \frac{f_1}{\nu}) \\
X'(\frac{f_{j+1}}{\nu}, x) &= X'(\frac{f_{j+1}-1}{\nu}, x) + X(\frac{i'_{j+1}}{\nu}, x) - X(\frac{i'_j}{\nu}, x) \\
X'(\frac{[t\nu]}{\nu}, x) &= X'(\frac{f_{j+1}}{\nu}, x) + X(\frac{i_{j+1}}{\nu} + \frac{[t\nu]-f_{j+1}}{\nu}, x) - X(\frac{i_{j+1}}{\nu}, x),
\end{aligned}$$

$$t \in [\frac{f_{j+1}+1}{\nu}, \frac{f_{j+2}}{\nu}), 0 \leq j \leq p-2.$$

$$X'(\frac{f_p}{\nu}, x) = X'(\frac{f_{p-1}}{\nu}, x) + X(\frac{i'_{p-1}+1}{\nu}, x) - X(\frac{i'_{p-1}}{\nu}, x)$$

$$X'(\frac{[t\nu]}{\nu}, x) = X'(\frac{f_p}{\nu}, x) \text{ if } t \in [\frac{f_p+1}{\nu}, 1)$$

We claim that  $X'$  is a nonstandard martingale, in the sense of Definition 0.7. In order to see this, suppose inductively that  $E_\eta(X'_{\frac{[t\nu]}{\nu}} | \mathcal{C}_\eta^{[\nu s]}) = X'_{\frac{[\nu s]}{\nu}}$ , for  $t \in [0, \frac{f_{j+1}}{\nu})$ , and  $s \leq t$ , where  $1 \leq j \leq p-3$ , then, for  $s' \leq \frac{f_{j+1}}{\nu}$ ;

$$\begin{aligned} E_\eta(X'_{\frac{f_{j+1}}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) &= E_\eta(X'_{\frac{f_{j+1}-1}{\nu}} + (X_{\frac{i'_{j+1}}{\nu}} - X_{\frac{i'_j}{\nu}}) | \mathcal{C}_\eta^{[\nu s']}) \\ &= X'_{\frac{[\nu s']}{\nu}} + (X_{\frac{[\nu s']}{\nu}} - X_{\frac{[\nu s']}{\nu}}) = X'_{\frac{[\nu s']}{\nu}} \quad (****) \end{aligned}$$

using the inductive assumption, (the case  $s' = \frac{f_{j+1}}{\nu}$  is clear), the fact that  $X$  is a nonstandard martingale, and  $\frac{[\nu s']}{\nu} \leq \frac{i'_j}{\nu}$ , as  $[\nu s'] \leq f_{j+1} \leq (j+1)\frac{\nu}{4p} \leq j[\frac{\nu}{2p}] = i_j \leq i'_j$ , by (!!) and (!!!).

$$\text{and, for } t \in [\frac{f_{j+1}+1}{\nu}, \frac{f_{j+2}}{\nu}), s' \leq t;$$

$$\begin{aligned} E_\eta(X'_{\frac{[t\nu]}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) &= E_\eta(X'_{\frac{f_{j+1}}{\nu}} + X_{\frac{i_{j+1}}{\nu} + \frac{[t\nu]-f_{j+1}}{\nu}} - X_{\frac{i_{j+1}}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) \\ &= X'_{\frac{[\nu s']}{\nu}} + (X_{\frac{[\nu s']}{\nu}} - X_{\frac{[\nu s']}{\nu}}) = X'_{\frac{[\nu s']}{\nu}} \end{aligned}$$

by (\*\*\*\*), the fact that  $X$  is a nonstandard martingale, and  $\frac{[\nu s']}{\nu} \leq \frac{i_{j+1}}{\nu}$ , as  $[\nu s'] \leq f_{j+2} \leq (j+2)\frac{\nu}{4p} \leq (j+1)[\frac{\nu}{2p}] = i_{j+1}$ , again, by (!!) and (!!!).

It follows that  $X'|_{[0, \frac{f_p}{\nu}) \times \overline{\Omega}_\eta}$  is a nonstandard martingale. Now, for  $s' \leq \frac{f_p}{\nu}$ ;

$$E_\eta(X'_{\frac{f_p}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) = X'_{\frac{[\nu s']}{\nu}}, (*****)$$

$$\text{as in } (****), \text{ and, for } t \in [\frac{f_p+1}{\nu}, 1), \frac{f_p}{\nu} \leq \frac{[\nu s']}{\nu} \leq t;$$

$$E_\eta(X'_{\frac{[t\nu]}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) = E_\eta(X'_{\frac{f_p}{\nu}} | \mathcal{C}_\eta^{[\nu s']}) = E_\eta(X'_{\frac{f_p}{\nu}} | \mathcal{C}_\eta^{f_p}) = X'_{\frac{f_p}{\nu}} = X'_{\frac{[\nu s']}{\nu}}$$

by (\*\*\*\*), the definition of  $X'$ , the fact that  $f_p \leq [\nu s']$  and  $X'_{\frac{f_p}{\nu}}$  is  $\mathcal{C}_\eta^{f_p}$ -measurable. The remaining case when  $\frac{[\nu s']}{\nu} < \frac{f_p}{\nu} \leq t$  is clear from (\*\*\*\*).

Let  $H'$  correspond to  $X'$  as in Definition 0.11, then, by the construction of  $X'$ ;

$$\int_{B \times \overline{\Omega}_\eta} H^2(t, x) d\lambda_\nu d\mu_\eta = \int_{[0, \frac{f_p-1}{\nu}] \times \overline{\Omega}_\eta} H^2(t, x) d\lambda_\nu d\mu_\eta \simeq 0$$

using the previous case, when  $B$  is an interval, applied to  $X'$ .

We now remove the assumptions (!!), (!!!). Suppose again that  $\text{Card}(I') = p$ , and  $\lambda_\nu([\frac{i_j}{\nu}, \frac{i'_j}{\nu})) \leq \frac{1}{4p}$ , for  $1 \leq j \leq p$ , (!!!), we obtain the condition (!!) as follows.

We define  $X'$  inductively, by;

$$X'(\frac{[t\nu]}{\nu}, x) = X(\frac{i_1+[t\nu]}{\nu}, x), t \in [0, \frac{f_1+1}{\nu})$$

$$X'(\frac{[t\nu]}{\nu}, x) = X'(\frac{f_1}{\nu}, x), t \in [\frac{f_1+1}{\nu}, \frac{[\frac{\nu}{2p}] - 1}{\nu})$$

$$X'(\frac{[\frac{\nu}{2p}] - 1}{\nu}, x) = X'(\frac{f_1}{\nu}, x) + (X(\frac{i_2}{\nu}, x) - X(\frac{i_2-1}{\nu}, x))$$

For  $1 \leq j \leq p-1$ ;

$$X'(\frac{[t\nu]}{\nu}, x) = X(\frac{i_{j+1}+[t\nu]}{\nu}, x), t \in [\frac{j}{\nu}[\frac{\nu}{2p}], \frac{j}{\nu}[\frac{\nu}{2p}] + \frac{(f_{j+1}-f_j+1)}{\nu})$$

$$X'(\frac{[t\nu]}{\nu}, x) = X'(\frac{j}{\nu}[\frac{\nu}{2p}] + \frac{(f_{j+1}-f_j)}{\nu}, x), t \in [\frac{j}{\nu}[\frac{\nu}{2p}] + \frac{(f_{j+1}-f_j+1)}{\nu}, \frac{(j+1)[\frac{\nu}{2p}] - 1}{\nu})$$

$$X'(\frac{(j+1)}{\nu}[\frac{\nu}{2p}] - \frac{1}{\nu}, x) = X'(\frac{j}{\nu}[\frac{\nu}{2p}] + \frac{(f_{j+1}-f_j)}{\nu}, x) + X(\frac{i_{j+2}}{\nu}, x) - X(\frac{i_{j+2}-1}{\nu}, x)$$

and;

$$X'(\frac{[t\nu]}{\nu}, x) = X'(\frac{p}{\nu}[\frac{\nu}{2p}] - \frac{1}{\nu}, x), t \in [\frac{p}{\nu}[\frac{\nu}{2p}], 1)$$

Similarly to the above,  $X'$  is a nonstandard martingale. By construction, we have found  $B'$ , satisfying (!!), (!!!) with;

$$\int_{B' \times \overline{\Omega}_\eta} H^2(t, x) d\lambda_\nu d\mu_\eta = \int_{B' \times \overline{\Omega}_\eta} H'^2(t, x) d\lambda_\nu d\mu_\eta \simeq 0$$

using the previous argument. We now show how to remove the assumption (!!!). Let  $B \subset \mathcal{T}_\nu$  be arbitrary, with  $\lambda_\nu(B) \simeq 0$ , and  $\text{Card}(I') = p$ . Let  $I'' = \{i_j \in I' : \lambda_\nu([\frac{i_j}{\nu}, \frac{i'_j}{\nu})) \geq \frac{1}{4p}\}$ , and  $B' = \bigcup_{i_j \in I''} [\frac{i_j}{\nu}, \frac{i'_j}{\nu})$ . It is easily seen that  $\text{Card}(I'') = q < p$ , for, otherwise,  $\lambda_\nu(B) = \lambda_\nu(B') \geq p \cdot \frac{1}{4p} = \frac{1}{4}$ , a contradiction. Let  $B'' = (B \setminus B')$ , so  $B = B'' \cup B'$ . For  $B''$ , with index  $I'''$ , we have  $\text{Card}(I''') = (p - q) < p$ , as if  $q = 0$ , we can use the assumption (!!!). Then adding  $q$  separated fibres of the form  $[\frac{t_j}{\nu}, \frac{t_{j+1}}{\nu}) \times \overline{\Omega}_\eta$ , ( $1 \leq j \leq q$ ), we can assume that  $B'' \subset B'''$ , with  $\text{Card}(I''') = p$ , and, again, the condition (!!!) applies to  $B'''$ . As  $\text{Card}(I'') = q < p$ , and  $\lambda_\nu(B') \leq \lambda_\nu(B)$ , we can use internal induction on  $\text{Card}(I'')$  to obtain the result.

Case 3. Suppose  $B \subset \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta$ , with  $(\lambda_\nu \times \mu_\eta)(B) \simeq 0$ , and, for  $t \in pr_\nu(B)$ ,  $\mu_\eta(pr_\eta(B \cap pr_\nu^{-1}(t))) \geq \frac{1}{4}$ . Then if  $B' = pr_\nu(B) \times \overline{\Omega}_\eta$ , we have  $(\lambda_\nu \times \mu_\eta)(B') \leq 4(\lambda_\nu \times \mu_\eta)(B) \simeq 0$ , and  $B \subset B'$ . Then, by Case 2;

$$0 \leq \int_B H^2(t, x) d\lambda_\nu d\mu_\eta \leq \int_{B'} H^2(t, x) d\lambda_\nu d\mu_\eta \simeq 0$$

Case 4. Suppose  $B \subset \overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta$ , with the property that for all  $t \in pr_\nu(B)$ ,  $(t + \frac{1}{\nu}) \notin pr_\nu(B)$ . We can easily make this assumption, by noting that  $B = B_1 \cup B_2$ , where  $B_1 = \{(\frac{i}{\nu}, x) \in B : 0 \leq i \leq \nu - 1, i \text{ odd}\}$   $B_2 = \{(\frac{i}{\nu}, x) \in B : 0 \leq i \leq \nu - 1, i \text{ even}\}$ , noting that  $\lambda_\nu \times \mu_\eta(B_1) \simeq 0$  and  $\lambda_\nu \times \mu_\eta(B_2) \simeq 0$ . By Case 3, we can also assume that, for all  $t \in pr_\nu(B)$ ,  $\mu_\eta(pr_\eta((B \cap pr_\nu^{-1}(t)))) \leq \frac{1}{4}$ , (!!!!). We let  $\omega_0$  be the least element of  $pr_\nu(B)$ . Define a subset  $B' \subset B$  to be joined, if  $B'$  is maximal with the property that  $B' = [\frac{j}{\nu}, \frac{j+1}{\nu}) \times [\frac{k_1}{2^j}, \frac{k_2}{2^j})$ , for some  $0 \leq j \leq \nu - 1$  and  $|k_2 - k_1| \geq 2$ . We enumerate these sets as  $B_{j, k_1, k_2}$ , for  $0 \leq j \leq \nu - 1$ ,  $0 \leq k_1 \leq k_2 \leq 2^j - 1$ . We first alter  $B$ , to remove all joined sets. We define  $g_0 : \mathcal{T}_\nu \times \overline{\Omega}_\eta \rightarrow \{0, 1\}$ , by requiring that  $g_0$  is  $\mathcal{D}_\nu \times \mathcal{C}_\eta$ -measurable and  $B = g_0^{-1}(0)$ . Let  $S_0 = \{(j, k, l) \in {}^*\mathcal{N}^3 : B_{j, k, l} \text{ joined}\}$ , and let  $(j_0, k_{1,0}, k_{2,0})$  be its least element, with the lexicographic ordering. Let  $Q_0 = \{y : 0 \leq y \leq 2^{j_0} - 1, (\frac{j_0}{\nu}, \frac{y-1}{2^{j_0}}), (\frac{j_0}{\nu}, \frac{y+1}{2^{j_0}}) \notin B\}$ , and  $\lambda_0 = \mu y (y \in Q_{0, j_0})$ .  $\lambda_0$  exists, as if  $Q_0 = \emptyset$ , then  $\mu_\eta(pr_\eta((B \cap pr_\nu^{-1}(t)))) \geq \frac{1}{3}$ , contradicting the assumption (!!!!). We inductively define a sequence  $\{(X_i, B_i) : 1 \leq i \leq 2^{\nu+1} - 2^{\omega_0}\}$  as follows. We first give an inductive definition of  $X_1$ ;

$$X_1(\frac{k}{\nu}, x) = X(\frac{k}{\nu}, x) \text{ for } 0 \leq k \leq (j_0 - 1)$$

$$X_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) = X\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right)$$

$$X_1\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) = X\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right)$$

$$X_1\left(\frac{j_0}{\nu}, \frac{\lambda_0+1}{2^{j_0}}\right) = 2X_1\left(\frac{j_0-1}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) - X_1\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) \text{ (assuming wlog } \lambda_0 \text{ is even)}$$

$$X_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}+1}{2^{j_0}}\right) = 2X_1\left(\frac{j_0-1}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) - X_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) \text{ (assuming wlog } k_{1,0} \text{ is even)}$$

$$X_1\left(\frac{j_0}{\nu}, \frac{k}{2^{j_0}}\right) = X\left(\frac{j_0}{\nu}, \frac{k}{2^{j_0}}\right) \text{ if } 0 \leq k \leq 2^{j_0}-1, k \notin \{\lambda_0, \lambda_0+1, k_{1,0}, k_{1,0}+1\}$$

Suppose  $X_{1,\frac{s}{\nu}}$  has been defined, for  $s \geq j_0$ , then set;

$$X_1\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}\right) = X_1\left(\frac{s}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right), 2^s k_{1,0} \leq t \leq 2^s(k_{1,0} + 1)$$

$$X_1\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0+1}}\right) = X_1\left(\frac{s}{\nu}, \frac{\lambda_0}{2^s}\right), 2^s \lambda_0 \leq t \leq 2^s(\lambda_0 + 1)$$

$$X_1\left(\frac{s+1}{\nu}, \frac{t}{2^s}\right) = X\left(\frac{s+1}{\nu}, \frac{t}{2^{s+1}}\right), t \notin [2^s k_{1,0}, 2^s(k_{1,0} + 1)] \cup [2^s \lambda_0, 2^s(\lambda_0 + 1)].$$

It is easily checked that  $X_1$  is a non standard martingale, in the sense of Definition 0.7, <sup>(10)</sup>. Similarly, we define  $B_1$ , by letting  $B_1 = g_1^{-1}(0)$ , where  $g_1$  is given inductively, by;

$$g_1\left(\frac{k}{\nu}, x\right) = g_0\left(\frac{k}{\nu}, x\right) \text{ for } 0 \leq k \leq (j_0 - 1)$$

$$g_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) = g_0\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right)$$

$$g_1\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) = g_0\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right)$$

$$g_1\left(\frac{j_0}{\nu}, \frac{\lambda_0+1}{2^{j_0}}\right) = 2g_1\left(\frac{j_0-1}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) - g_1\left(\frac{j_0}{\nu}, \frac{\lambda_0}{2^{j_0}}\right) \text{ (assuming wlog } \lambda_0 \text{ is even)}$$

$$g_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}+1}{2^{j_0}}\right) = 2g_1\left(\frac{j_0-1}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) - g_1\left(\frac{j_0}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right) \text{ (assuming wlog } k_{1,0} \text{ is even)}$$

$$g_1\left(\frac{j_0}{\nu}, \frac{k}{2^{j_0}}\right) = g_0\left(\frac{j_0}{\nu}, \frac{k}{2^{j_0}}\right) \text{ if } 0 \leq k \leq 2^{j_0}-1, k \notin \{\lambda_0, \lambda_0+1, k_{1,0}, k_{1,0}+1\}$$

Suppose  $g_{1,\frac{s}{\nu}}$  has been defined, for  $s \geq j_0$ , then set;

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<sup>10</sup>We have used the criteria  $\int_{C_j} X_{\frac{i}{\nu}} d\mu_\eta = \int_{C_j} X_{\frac{i+1}{\nu}} d\mu_\eta$ , for an irreducible interval in the filtration  $\mathcal{C}_\eta^i$

$$g_1\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}\right) = g_1\left(\frac{s}{\nu}, \frac{k_{1,0}}{2^{j_0}}\right), 2^s k_{1,0} \leq t \leq 2^s(k_{1,0} + 1)$$

$$g_1\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}\right) = g_1\left(\frac{s}{\nu}, \frac{\lambda_0}{2^{j_0}}\right), 2^s \lambda_0 \leq t \leq 2^s(\lambda_0 + 1)$$

$$g_1\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}\right) = g_0\left(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}\right), t \notin [2^s k_{1,0}, 2^s(k_{1,0} + 1)] \cup [2^s \lambda_0, 2^s(\lambda_0 + 1)].$$

Now suppose we have defined  $(X_i, B_i)$ , for  $0 \leq i < 2^{\nu+1} - 2^{\omega_0}$ . As before, we define  $\{S_i, Q_i, \lambda_i, j_i, k_{1,i}, k_{2,i}, g_i\}$ . If  $B_i$  still contains joined sets, with respect to the nonstandard martingale  $X_i$ , then we define  $X_{i+1}$  by;

$$X_{i+1}\left(\frac{k}{\nu}, x\right) = X_i\left(\frac{k}{\nu}, x\right) \text{ for } 0 \leq k \leq (j_i - 1)$$

$$X_{i+1}\left(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right) = X_i\left(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}\right)$$

$$X_{i+1}\left(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}\right) = X_i\left(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right)$$

$$X_{i+1}\left(\frac{j_i}{\nu}, \frac{\lambda_i+1}{2^{j_i}}\right) = 2X_{i+1}\left(\frac{j_i-1}{\nu}, \frac{\lambda_i}{2^{j_i}}\right) - X_{i+1}\left(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}\right) \text{ (assuming wlog } \lambda_i \text{ is even)}$$

$$X_{i+1}\left(\frac{j_i}{\nu}, \frac{k_{1,i}+1}{2^{j_i}}\right) = 2X_{i+1}\left(\frac{j_i-1}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right) - X_{i+1}\left(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right) \text{ (assuming wlog } k_{1,i} \text{ is even)}$$

$$X_{i+1}\left(\frac{j_i}{\nu}, \frac{k}{2^{j_i}}\right) = X_i\left(\frac{j_i}{\nu}, \frac{k}{2^{j_i}}\right) \text{ if } 0 \leq k \leq 2^{j_i} - 1, k \notin \{\lambda_i, \lambda_i + 1, k_{1,i}, k_{1,i} + 1\}$$

Suppose  $X_{i+1, \frac{s}{\nu}}$  has been defined, for  $s \geq j_i$ , then set;

$$X_{i+1}\left(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}\right) = X_{i+1}\left(\frac{s}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right), 2^s k_{1,i} \leq t \leq 2^s(k_{1,i} + 1)$$

$$X_{i+1}\left(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}\right) = X_{i+1}\left(\frac{s}{\nu}, \frac{\lambda_i}{2^{j_i}}\right), 2^s \lambda_i \leq t \leq 2^s(\lambda_i + 1)$$

$$X_{i+1}\left(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}\right) = X_i\left(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}\right), t \notin [2^s k_{1,i}, 2^s(k_{1,i} + 1)] \cup [2^s \lambda_i, 2^s(\lambda_i + 1)].$$

Similarly, we define  $B_{i+1}$ , by letting  $B_{i+1} = g_{i+1}^{-1}(0)$ , where  $g_{i+1}$  is given inductively, by;

$$g_{i+1}\left(\frac{k}{\nu}, x\right) = g_i\left(\frac{k}{\nu}, x\right) \text{ for } 0 \leq k \leq (j_i - 1)$$

$$g_{i+1}\left(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}}\right) = g_i\left(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}\right)$$

$$g_{i+1}(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}) = g_i(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}})$$

$$g_{i+1}(\frac{j_i}{\nu}, \frac{\lambda_i+1}{2^{j_i}}) = 2g_{i+1}(\frac{j_i-1}{\nu}, \frac{\lambda_i}{2^{j_i}}) - g_{i+1}(\frac{j_i}{\nu}, \frac{\lambda_i}{2^{j_i}}) \text{ (assuming wlog } \lambda_i \text{ is even)}$$

$$g_{i+1}(\frac{j_i}{\nu}, \frac{k_{1,i}+1}{2^{j_i}}) = 2g_{i+1}(\frac{j_i-1}{\nu}, \frac{k_{1,i}}{2^{j_i}}) - g_{i+1}(\frac{j_i}{\nu}, \frac{k_{1,i}}{2^{j_i}}) \text{ (assuming wlog } k_{1,i} \text{ is even)}$$

$$g_{i+1}(\frac{j_i}{\nu}, \frac{k}{2^{j_i}}) = g_i(\frac{j_i}{\nu}, \frac{k}{2^{j_i}}) \text{ if } 0 \leq k \leq 2^{j_i}-1, k \notin \{\lambda_i, \lambda_i+1, k_{1,i}, k_{1,i}+1\}$$

Suppose  $g_{i+1, \frac{s}{\nu}}$  has been defined, for  $s \geq j_i$ , then set;

$$g_{i+1}(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}) = g_{i+1}(\frac{s}{\nu}, \frac{k_{1,i}}{2^{j_i}}), 2^s k_{1,i} \leq t \leq 2^s(k_{1,i}+1)$$

$$g_{i+1}(\frac{s+1}{\nu}, \frac{t}{2^{j_i}}) = g_{i+1}(\frac{s}{\nu}, \frac{\lambda_i}{2^{j_i}}), 2^s \lambda_i \leq t \leq 2^s(\lambda_i+1)$$

$$g_{i+1}(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}) = g_i(\frac{s+1}{\nu}, \frac{t}{2^{j_0}}), t \notin [2^s k_{1,i}, 2^s(k_{1,i}+1)] \cup [2^s \lambda_i, 2^s(\lambda_i+1)].$$

Clearly, this process terminates in at most  $\sum_{j=0}^{\nu(1-\omega_0)-1} 2^{\omega_0 \nu + j} = 2^{\nu \omega_0} (2^{\nu(1-\omega_0)} - 1) = \kappa$  steps, and we obtain a nonstandard martingale  $(X_\kappa, B_\kappa)$ , with  $(\lambda_\nu \times \mu_\eta)(B_\kappa) = (\lambda_\nu \times \mu_\eta)(B) \simeq 0$ , and;

$$\int_B H^2(t, x) d(\lambda_\nu \times \mu_\eta) = \int_{B_\kappa} H_\kappa^2(t, x) d(\lambda_\nu \times \mu_\eta)$$

We can therefore reduce to;

Case 5. Assume  $(X, B)$  satisfies the hypotheses of Case 4, with the additional property that  $B$  has no joined subsets, (\*\*\*\*). We reduce to Case 1. We define a component  $B' \subset B$  to be of the form  $B' = [\frac{j}{\nu}, \frac{j+1}{\nu}] \times [\frac{l_1}{2^j}, \frac{l_1+1}{2^j}]$ , for some  $0 \leq j \leq \nu-1$ . Enumerating these sets as  $B_{j, l_1}$ , for  $0 \leq j \leq \nu-1, 0 \leq l_1 \leq 2^j-1$ , we let  $S_0 = \{(j, l_1) \in \mathcal{N}^2, B_{j, l_1} \text{ is a component}\}$ , as  $B_{j, l_1}$ . Observe that, by the assumption (\*\*\*\*),  $B = \bigcup_{(j, l_1) \in S_0} B_{j, l_1}$ . Let  $(\omega_0, l_{1,0})$  denote the least element of  $S_0$ . We also assume that there exists  $\epsilon \simeq 0$ , such that, for all  $v \geq \omega_0$ ,  $\mu_\eta(B \cap [\frac{v}{\nu}, \frac{v+1}{\nu}]) \leq \epsilon$ , (#####). Then we define a sequence of nonstandard martingales  $(X_j, B_j)$ . We define  $X_1$ , inductively, by constructing a sequence  $(X_{1,i}, B_{1,i})$  as follows. Suppose  $(X_{1,i}, B_{1,i})$  has been defined. Let  $y_{i+1} = \mu y (\omega_0 + 1 \leq y \leq \nu-1)$ , such that  $T_{i+1,1,y} = ([\frac{y}{\nu}, \frac{y+1}{\nu}] \times [\frac{l_{1,0}}{2^{\omega_0}}, \frac{l_{1,0}+1}{2^{\omega_0}}]) \cap B_{1,i}^c \neq \emptyset, ([\frac{y}{\nu}, \frac{y+1}{\nu}] \times [\frac{l_{1,0}}{2^{\omega_0}}, \frac{l_{1,0}+1}{2^{\omega_0}}])^{comp} \cap B_{1,i} \neq \emptyset$ , and there exists  $\frac{q}{2^y}$ , with  $\{\frac{q-1}{2^y}, \frac{q}{2^y}, \frac{q+1}{2^y}\} \subset T_{i+1,1,y}$ . Let  $T_{i+1,1} = ([\frac{y_{i+1}}{\nu}, \frac{y_{i+1}+1}{\nu}] \times$



$[\frac{l_{1,0}}{2^{\omega_0}}, \frac{l_{1,0}+1}{2^{\omega_0}}]) \cap B_{1,i}^c$  and  $W_{i+1,1} = ([\frac{y_{i+1}}{\nu}, \frac{y_{i+1}+1}{\nu}) \times [\frac{l_{1,0}}{2^{\omega_0}}, \frac{l_{1,0}+1}{2^{\omega_0}})^{comp}) \cap B_{1,i}$ .  
 Let  $\theta_{i+1} = \mu z((\frac{y_{i+1}}{\nu}, \frac{z}{2^{y_{i+1}}}) \in T_{i+1,1}, \{\frac{z-1}{2^{y_{i+1}}}, \frac{z}{2^{y_{i+1}}}, \frac{z+1}{2^{y_{i+1}}}\} \subset T_{i+1,1})$  and  $z_{i+1} = \mu z((\frac{y_{i+1}}{\nu}, \frac{z}{2^{y_{i+1}}}) \in W_{i+1,1}$ . Define  $X_{1,i+1}$  by;

$$X_{1,i+1}(\frac{j}{\nu}, x) = X_{1,i}(\frac{j}{\nu}, x) \text{ for } 0 \leq j \leq y_{i+1} - 1$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}}) = X_{1,i}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}})$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}+1}{2^{y_{i+1}}}) = 2X_{1,i+1}(\frac{y_{i+1}-1}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}-1}}) - X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}})$$

(if  $\theta_{i+1}$  is even)

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}-1}{2^{y_{i+1}}}) = 2X_{1,i+1}(\frac{y_{i+1}-1}{\nu}, \frac{\theta_{i+1}-1}{2^{y_{i+1}-1}}) - X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}}) \text{ (if } \theta_{i+1} \text{ is odd)}$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}}) = X_{1,i}(\frac{y_{i+1}}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}})$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}+1}{2^{y_{i+1}}}) = 2X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}}) - X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}}) \text{ (if } z_{i+1} \text{ is even)}$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}-1}{2^{y_{i+1}}}) = 2X_{1,i+1}(\frac{y_{i+1}-1}{\nu}, \frac{z_{i+1}-1}{2^{y_{i+1}-1}}) - X_{1,i+1}(\frac{y_{i+1}}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}}) \text{ (if } z_{i+1} \text{ is odd)}$$

$$X_{1,i+1}(\frac{y_{i+1}}{\nu}, x) = X_{1,i+1}(\frac{y_{i+1}}{\nu}, x), x \notin [\frac{z_{i+1}}{2^{y_{i+1}}}, \frac{z_{i+1}+1}{2^{y_{i+1}}}] \cup [\frac{\theta_{i+1}}{2^{y_{i+1}}}, \frac{\theta_{i+1}+1}{2^{y_{i+1}}}]$$

(-1 for odd case)

Suppose  $X_{1,i+1}$  has been defined on  $[0, \frac{s}{\nu}) \times \overline{\Omega_\eta}$ , for  $s > y_{i+1}$ , then let;

$$X_{1,i+1}(\frac{s+1}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}} + \frac{k}{2^{s+1}}) = X_{1,i}(\frac{s+1}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}} + \frac{k}{2^{s+1}}), 0 \leq k \leq 2^{1+s-y_{i+1}} - 1$$

$$X_{1,i+1}(\frac{s+1}{\nu}, \frac{\theta_{i+1}}{2^{y_{i+1}}} + \frac{k}{2^{s+1}}) = X_{1,i}(\frac{s+1}{\nu}, \frac{z_{i+1}}{2^{y_{i+1}}} + \frac{k}{2^{s+1}}), 0 \leq k \leq 2^{1+s-y_{i+1}} - 1$$

$$X_{1,i+1}(\frac{s+1}{\nu}, \frac{k}{2^{s+1}}) = X_{1,i}(\frac{s+1}{\nu}, \frac{k}{2^{s+1}}), \frac{k}{2^{s+1}} \notin [\frac{\theta_{i+1}}{2^{y_{i+1}}}, \frac{\theta_{i+1}+1}{2^{y_{i+1}}}] \cup [\frac{z_{i+1}}{2^{y_{i+1}}}, \frac{z_{i+1}+1}{2^{y_{i+1}}})$$

Clearly this process terminates in at most;

$$\kappa \leq * \sum_{j=0}^{\nu-1-\omega_0} (2^{\omega_0+j} - 2^j) = (2^{\omega_0} - 1)(2^{\nu-\omega_0} - 1)$$

steps. We set  $(X_1, B_1) = (X_{1,\kappa}, B_{1,\kappa})$ . Suppose we have constructed  $(X_j, B_j)$ ., We define  $X_{j+1}$ , inductively, by constructing a sequence  $(X_{j+1,i}, B_{j+1,i})$  as follows. Suppose  $(X_{j+1,i}, B_{j+1,i})$  has been defined. Let  $y_{j+1,i+1} = \mu y(\omega_j + 1 \leq y \leq \nu - 1)$ , such that  $T_{i+1,j+1,y} = ((\frac{y}{\nu}, \frac{y+1}{\nu}) \times$

$([\frac{l_{j+1}}{2^{\omega_j}}, \frac{l_{j+1}+1}{2^{\omega_j}}]) \cap B_{j+1,i}^c \neq \emptyset, ([\frac{y}{\nu}, \frac{y+1}{\nu}] \times [\frac{l_{j+1}}{2^{\omega_j}}, \frac{l_{j+1}+1}{2^{\omega_j}}]^{comp}) \cap B_{j+1,i} \neq \emptyset$ ,  
and there exists  $\frac{q}{2^y}$ , with  $\{\frac{q-1}{2^y}, \frac{q}{2^y}, \frac{q+1}{2^y}\} \subset T_{i+1,j+1,y}$ . Let  $T_{i+1,j+1} =$   
 $([\frac{y_{j+1,i+1}}{\nu}, \frac{y_{j+1,i+1}+1}{\nu}] \times [\frac{l_{j+1}}{2^{\omega_j}}, \frac{l_{j+1}+1}{2^{\omega_j}}]) \cap B_{j+1,i}^c$  and  $W_{i+1,j+1} = ([\frac{y_{j+1,i+1}}{\nu}, \frac{y_{j+1,i+1}+1}{\nu}] \times$   
 $[\frac{l_{j+1}}{2^{\omega_j}}, \frac{l_{j+1}+1}{2^{\omega_j}}]^{comp}) \cap B_{j+1,i}$ . Let  $\theta_{j+1,i+1} = \mu z((\frac{y_{j+1,i+1}}{\nu}, \frac{z}{2^{y_{j+1,i+1}}}) \in$   
 $T_{i+1,j+1}, \{\frac{z-1}{2^{y_{j+1,i+1}}}, \frac{z}{2^{y_{j+1,i+1}}}, \frac{z+1}{2^{y_{j+1,i+1}}}\} \subset T_{i+1,j+1})$  and  $z_{j+1,i+1} = \mu z((\frac{y_{j+1,i+1}}{\nu}, \frac{z}{2^{y_{j+1,i+1}}}) \in$   
 $W_{i+1,j+1}$ . Define  $X_{j+1,i+1}$  by;

$$X_{j+1,i+1}(\frac{a}{\nu}, x) = X_{j+1,i}(\frac{a}{\nu}, x) \text{ for } 0 \leq a \leq y_{j+1,i+1} - 1$$

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}}) = X_{j+1,i}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}+1}{2^{y_{j+1,i+1}}}) = 2X_{j+1,i+1}(\frac{y_{j+1,i+1}-1}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}-1}}) - X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

(if  $\theta_{j+1,i+1}$  is even)

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}-1}{2^{y_{j+1,i+1}}}) = 2X_{j+1,i+1}(\frac{y_{j+1,i+1}-1}{\nu}, \frac{\theta_{j+1,i+1}-1}{2^{y_{j+1,i+1}-1}}) - X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

(if  $\theta_{j+1,i+1}$  is odd)

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}}) = X_{j+1,i}(\frac{y_{j+1,i+1}}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}+1}{2^{y_{j+1,i+1}}}) = 2X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}}) - X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

(if  $z_{j+1,i+1}$  is even)

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}-1}{2^{y_{j+1,i+1}}}) = 2X_{j+1,i+1}(\frac{y_{j+1,i+1}-1}{\nu}, \frac{z_{j+1,i+1}-1}{2^{y_{j+1,i+1}-1}}) - X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}})$$

(if  $z_{j+1,i+1}$  is odd)

$$X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, x) = X_{j+1,i+1}(\frac{y_{j+1,i+1}}{\nu}, x), x \notin [\frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}}, \frac{z_{j+1,i+1}+1}{2^{y_{j+1,i+1}}}] \cup$$

$$[\frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}}, \frac{\theta_{j+1,i+1}+1}{2^{y_{j+1,i+1}}}]$$

(-1 for odd case)

Suppose  $X_{j+1,i+1}$  has been defined on  $[0, \frac{s}{\nu}] \times \overline{\Omega_\eta}$ , for  $s > y_{i+1}$ , then let;

$$X_{j+1,i+1}(\frac{s+1}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}} + \frac{k}{2^{s+1}}) = X_{j+1,i}(\frac{s+1}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}} + \frac{k}{2^{s+1}}), 0 \leq k \leq$$

$$2^{1+s-y_{j+1,i+1}} - 1$$

$$X_{j+1,i+1}(\frac{s+1}{\nu}, \frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}} + \frac{k}{2^{s+1}}) = X_{j+1,i}(\frac{s+1}{\nu}, \frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}} + \frac{k}{2^{s+1}}), 0 \leq k \leq$$

$$2^{1+s-y_{j+1,i+1}} - 1$$

$$X_{j+1,i+1}(\frac{s+1}{\nu}, \frac{k}{2^{s+1}}) = X_{j+1,i}(\frac{s+1}{\nu}, \frac{k}{2^{s+1}}), \frac{k}{2^{s+1}} \notin [\frac{\theta_{j+1,i+1}}{2^{y_{j+1,i+1}}}, \frac{\theta_{j+1,i+1}+1}{2^{y_{j+1,i+1}}} \cup [\frac{z_{j+1,i+1}}{2^{y_{j+1,i+1}}}, \frac{z_{j+1,i+1}+1}{2^{y_{j+1,i+1}}})$$

Clearly this process terminates in at most;

$$\kappa \leq \sum_{k=0}^{\nu-1-\omega_j} (2^{\omega_j+k} - 2^k) = (2^{\omega_j} - 1)(2^{\nu-\omega_j} - 1)$$

steps. We set  $(X_{j+1}, B_{j+1}) = (X_{j+1,\kappa}, B_{j+1,\kappa})$ . We then obtain a terminating sequence of non-standard martingales  $\{X_j, B_j : 1 \leq j \leq \rho\}$  ( $\rho?$ ). Taking  $(Y, B') = (X_\rho, B_\rho)$ , with corresponding  $H_Y$ , we have that;

$$\int_{\overline{\Omega}_\eta \times \overline{T}_\nu} H_Y^2(t, x) d(\lambda_\nu \times \mu_\eta) = \int_{\overline{\omega}_\eta \times \overline{T}_\nu} H_X^2(t, x) d(\lambda_\nu \times \mu_\eta)$$

and  $B' \subset ([\frac{\omega_0}{\nu}, \frac{\nu-1}{\nu}) \times D) \subset (\mathcal{T}_\nu \times D)$ , with  $\mu_\eta(D) \leq \epsilon \simeq 0$ , by (#####). Using Case 1, we obtain that;

$$\int_{\overline{\Omega}_\eta \times \overline{T}_\nu} H_X(t, x) d(\lambda_\nu \times \mu_\eta) \simeq 0.$$

Case 6. Let  $(X, B)$  be arbitrary, with  $\lambda_\nu \times \mu_\eta(B) = \delta \simeq 0$ . Let  $c = \text{Card}(\{j : \mu_\eta(\text{pr}_\eta(B \cap [\frac{\omega_0+j}{\nu}, \frac{\omega_0+j+1}{\nu}))) > \sqrt{\delta}\})$ , and  $A_{\delta,c} = \{\frac{\omega_0+j}{\nu} : \mu_\eta(\text{pr}_\eta(B \cap [\frac{\omega_0+j}{\nu}, \frac{\omega_0+j+1}{\nu}))) > \sqrt{\delta}\}$ . Let  $B_{\delta,c} = \bigcup_{t \in A_{\delta,c}} [t, t + \frac{1}{\nu})$ . Then  $B = (B \cap (B_{\delta,c} \times \overline{\Omega}_\eta)) \cup (B \cap (B_{\delta,c} \times \overline{\Omega}_\eta)^c) = B_3 \cup B_4$ . We have  $\delta \geq (\lambda_\nu \times \mu_\eta)(B_1) \geq (\delta^{\frac{1}{2}})_{\nu}^c$ , so  $\lambda_\nu(B_3) = \frac{c}{\nu} \leq \frac{\delta}{\delta^{\frac{1}{2}}} = \delta^{\frac{1}{2}} \simeq 0$ . By Case 2, we have that;

$$\int_{B_3} H^2(t, x) d(\lambda_\nu \times \mu_\eta) \simeq 0$$

By Case 4,5, (#####), using the fact that  $\mu_\eta(\text{pr}_\eta(B_4 \cap \text{pr}_\nu^{-1}(t))) \leq \delta^{\frac{1}{2}} \simeq 0$ , for  $t \in \mathcal{T}_\nu$ , we have that;

$$\int_{B_4} H^2(t, x) d(\lambda_\nu \times \mu_\eta) \simeq 0$$

as required.

**Definition 0.13.** We define a martingale with independent increments, to be a function  $I : [0, 1] \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R}$  satisfying footnote 2, with the additional requirement (vi) that, for  $0 \leq t_1 < t_2 \leq t_3 < t_4 < 1$ ;

$$I_{t_4} - I_{t_3} \text{ is independent of } I_{t_2} - I_{t_1}$$

**Theorem 0.14.** *Any martingale  $X$  with independent increments is representable as a stochastic integral;*

$$X(t, x) = \int_0^t F(s, x) dW_s$$

where  $F : [0, 1] \times \overline{\Omega}_\eta \rightarrow {}^*\mathcal{R} \in L^2([0, 1] \times \overline{\Omega}_\eta, L(\mu_\eta))$ .

*Proof.* By Lemma 0.10, there exists a nonstandard martingale  $\overline{X}$ , with  ${}^\circ(\overline{X}_t) = X_{\circ t}$ , for  $t \in \overline{\mathcal{T}}_\nu$ , a.e  $L(\mu_\eta)$ . By Lemma 0.8, we have the representation;

$$\overline{X}_t(x) = \sum_{j=0}^{[\nu t]} c_j(t, x) \omega_j(x)$$

Letting  $H(t, x) = (\sqrt{\nu})c_j(1, x)$ , as in Definition 0.11, we set  $h_j(x) = ({}^\circ\sqrt{\nu}c_j)(1, x) = H(1, x)$ , and  $d(s, x) = h_{\lfloor \frac{\nu s}{\nu} \rfloor}(x)$ . We set  $Z(t, x) = \int_0^t d(s, x) dW_s$ , and claim  $X = Z$  as stochastic processes. We first show that  $H$  is a 2-lifting of  $d$ , by verifying Conditions (i)-(iii) in Definition 30 of [1], see also Definition 7.18 of [3].

$$(i). H \in SL^2(\overline{\mathcal{T}}_\nu \times \overline{\Omega}_\eta, \lambda_\nu \times \mu_\eta)$$

by Lemma 0.12.

$$(ii). {}^\circ H(t, x) = d({}^\circ t, x)$$

$$\text{We have } {}^\circ H(t, x) = ({}^\circ(\sqrt{\nu}c_j)(1, x))$$

$$\text{where } j = \frac{[t\nu]}{\nu}.$$

$$= {}^\circ h_{\lfloor \frac{\nu t}{\nu} \rfloor}(x) = d({}^\circ t, x)$$

$$(iii). H(t, x) \text{ is } \mathcal{C}_\eta^{[\nu t]} \text{ measurable.}$$

by Definition 0.11 and Lemma 0.8.

Then  $X = Z$  are equivalent as stochastic processes by Theorem 33 of [1], see also Theorem 7.22 of [3].

□

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REFERENCES

- [1] A Non-Standard Representation for Brownian Motion and Ito Integration, Robert Anderson, Isreal Journal of Mathematics, (1976).
- [2] Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory, Peter Loeb, Transactions of the American Mathematical Society, (1975).
- [3] Applications of Nonstandard Analysis to Probability Theory, Tristram de Piro, M.Sc Dissertation in Financial Mathematics, University of Exeter, (2013).
- [4] Real and Complex Analysis, Walter Rudin, McGraw Hill Third Edition, (1987).
- [5] Stochastic Calculus and Financial Applications, Michael Steele, Applications of Mathematics, Springer, (2001).
- [6] Probability with Martingales, David Williams, Cambridge Mathematical Textbooks, (1991).

MATHEMATICS DEPARTMENT, HARRISON BUILDING, STREATHAM CAMPUS,  
UNIVERSITY OF EXETER, NORTH PARK ROAD, EXETER, DEVON, EX4 4QF,  
UNITED KINGDOM

*E-mail address:* `tdpd201@exeter.ac.uk`